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Oscillation Criteria for the Solution of Second Order Ordinary Linear Differential Equations with Integrable Coefficient

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ABSTRACT

This study considers condition of oscillation for solution of second order linear differential equations using an existing technique name integral technique. It is found that the technique gives the criteria of oscillation of equations been investigated. Therefore, it is recommended for further investigation a case when $n > 2$ using the same techniques as above.

Keywords: Wong Technique, Oscillation; Differential Equations, and Linear Equation

INTRODUCTION

Ordinary Differential Equations (ODE) are use to describe many phenomena of physical interest, because of their frequent use in the investigation of a wide variety of problems in the field of science, such as transforming mechanical problems to system of differential equations (Modeling) in Engineering, principles in Physics, fluid dynamics such as measuring the rate of flow or concentration of impurities in a fluid tank, in Industrial Chemistry and modeling the dynamics of population (demography). Ordinary Differential Equation (ODE) contain derivatives of which depend on the solution at the present value of the independent variable that is time [1].

[1] investigated the oscillation criteria of second order ordinary linear differential equation of the form

$$y^{(n)}(t) + q(t)y(t) = 0 \quad t \geq t_0 \qquad (1)$$

Where

$q \in C([t_0, \infty))$

A solution $y(t)$ of equation (1) is oscillatory if it is neither eventually positive nor eventually negative. If there exist an eventually positive or eventually negative solution, the equation is non-oscillatory. Also equation (1) is said to be oscillatory if its solutions have arbitrarily large zeros on the half range, The consequence of Sturm's Separation theorem state that, if one of the solutions of equation (1) is oscillatory, then all of remaining solution are oscillatory, the converse is also true. Also Sturm comparison theorem entails the comparison of equation (1) with another second order linear differential equation, of the same form, then oscillation or non-oscillation behavior of which is used to determine that of equation (1).

[1], investigating this problem used the Euler's classical equation as the basic of comparison and considered $q(t) = \frac{\sin \beta t}{t}$ in (1.1). In this work our choice of $q(t) = \frac{\sin \beta t}{t^{\lambda}}$ in (1) will be in line with that of Wong [1] if $\lambda = 1$, but our case deal with $\lambda > 1$

[2], studied the significance of the integral condition for $q(t) \geq 0$ with $\int_t^{\infty} q(s) = \infty$ and pointed out no individual t-value occur explicitly in the condition. on oscillation condition of differential equation, and contributed to the comparison studies of the oscillation and non-oscillation of equation. [1] showed that classical result of Sturm gives the comparison criteria regarding the oscillation of differential equation of form in equation (1) when

compared with another differential equation of the same form. Also classical theorem of Sturm's assert that if $q(t) > o$ in equation (1) implies oscillation and otherwise, non-oscillation.

[3] investigated on separation of root and oscillation in ordinary linear differential equation of second order, found out that the location of their root are obtained with the aid of a new type Sturm comparison equation.

[4] categorized the oscillation of equation (1.1) into distinctive cases all of which employed integration of $q(t)$ in equation. The first case is under the assumptions that $\lim_{t\to\infty}\int_t^\infty q(s)ds,$ exist and is finite. The second case consists of equation for which $\lim_{t\to\infty}\int_t^\infty q(s)ds,$ tend to -∞. However, they noted that this simple classification is unsatisfactory since many equations that do not satisfy the additional hypothesis such as $q(t)$ being bounded on one side as required second case are still unclassified.

[5] investigated oscillation theorems for second order sub-linear differential equation, and found out oscillation criterion is given for the second order sub-linear differential equations $x''+a(t)|X|^y \operatorname{sgn} x = 0$, $0 < y < 1$, where the coefficient a(t) is not assumed to be non-negative for all large values of t.

As part of an important application of oscillation second order linear differential equation, we consider the work of [6] on the oscillation theory of differential equations, proposed a modified mathematical model for the onset of large amplitude oscillation in suspension bridges forced by wind with specific velocities. In the model the motion of the bridge is usual governed by a system of differential equations. One of the ideas introduced has to do with the symmetry of the restoring force when a cable extends under expansion or compression, basic assumption was that the cable " strongly resisted expansion but did not resist compression" their study led to second order ordinary differential equation with periodic boundary conditions.

[7] investigated Wong-type oscillation theorem for second order linear dynamic equations on time scales to obtain the Wong-type oscillation theorems for second order linear dynamic equations on a time scale. The results obtained are motivated by oscillation results due to Wong. As a particular application of the results, it is shown that the difference equation $\Delta^2 x(n) + \frac{b(-1)^n}{n} x(x+1) = 0$ is oscillatory, if and only if $|b|>1$.

[9] studied oscillation criteria for Sturm-Lioville operator coefficient in the limit circle case, and found out good criteria for oscillation of solution u to Sturm-Liouville equation $(L-\lambda)u = 0$. In particular, no necessary and sufficient efficient conditions on the co-efficients of L are known.

METHODOLOGY

We use the integral techniques of [1] to investigation the criteria of oscillation of second order ordinary linear differential equation of (1)

[1] Integral Technique

If $Q(t)$ exist and $Q(t)$ is bounded that is

$$\lim_{t\to\infty}\int_0^t q(s)ds < \infty \qquad (2)$$

where

$$Q(t) = \int_t^\infty q(s)ds \qquad (3)$$

The conjugate of $Q(t)$ is given as

$$\int_t^\infty \overline{Q}^2(s)\, P_Q(s,t)ds > \frac{1}{4}\overline{Q}(t) \qquad (4)$$

where

$$P_Q(s,t) = \exp(2\int_t^s Q(\tau)d\tau) \qquad (5)$$

Satisfies

$$\int_0^\infty \exp(-4\int_0^t \overline{Q}(s)ds)dt < \infty \quad (6)$$

where

$$\overline{Q}(t) = \int_t^\infty Q^2(s)P_Q(s,t)ds \quad (7)$$

Then equation (1) is oscillatory.
Similarly if

$$\int_t^\infty Q^2(s)\exp(2\int_t^\infty Q(\tau)d\tau)dt = \infty \quad (8)$$

and

$$\int_t^\infty \overline{Q}(s)\exp(2\int_t^\infty \overline{Q}(\tau)d\tau)dt = \infty \quad (9)$$

[1] assumes basically that t takes a very large value and he also showed that if

$$\int_t^\infty \overline{Q}^2(s)P_Q(s,t)ds > \frac{1}{4}\overline{Q}(t) \quad (10)$$

is oscillatory, then

$$\left|\frac{1}{\beta}\right| > \frac{1}{\sqrt{2}}$$

and also if

$$\int_t^\infty \overline{Q}^2(s)P_Q(s,t)ds < \frac{1}{4}\overline{Q}(t) \quad (11)$$

is non-oscillation, then

$$\left|\frac{1}{\beta}\right| < \frac{1}{\sqrt{2}}$$

Wong's technique will also be used to obtain criteria for oscillation of the same differential equation.

RESULTS

The [1] integral techniques would be used to determine the oscillation of second order linear differential equations

Oscillation Criteria by Integral Techniques

Consider the equation below

$$y^{(n)}(t) + q(t)y(t) = 0 \quad (12)$$

Consider the case where $n = 2$

If

$$Q(t) = \int_t^\infty q(s)ds \quad (13)$$

and

$$\overline{Q}(t) = \int_t^\infty Q^2(s)P_Q(s,t)ds \quad (14)$$

where

$$P_Q(s,t) = \exp(2\int_t^\infty Q(\tau)d\tau) \quad (15)$$

If

$$\int_t^\infty \overline{Q}^2(s)P_Q(s,t)ds > \frac{1}{4}\overline{Q}(t) \quad (16)$$

Then

$Q(t)$ is oscillatory

Consider the case where $q(t) = \frac{\sin\beta t}{t}$ in equation (12).

Now from (3)

$$Q(t) = \int_t^\infty \frac{\sin\beta s}{s}ds = \frac{\cos\beta t}{\beta t} + 0\left(\frac{1}{t^2}\right). \quad (17)$$

From (4) we have

$$P_Q(s,t) = 1 + 2\left(\int_s^s \frac{\cos\beta\tau}{\beta\tau} + 0\left(\frac{1}{t^2}\right)\right)d\tau + \ldots$$

$$= 1 + 0\left(\frac{1}{t^2}\right). \quad (18)$$

Thus from (17)

$$Q^2(t) = \frac{\cos^2\beta t}{\beta^2 t^2}. \quad (19)$$

Substituting (19) and (18) into (14) we have

$$\overline{Q}(t) = \int_{tt}^\infty Q^2(s)P_q(s,t)ds = \int_t^\infty \frac{\cos^2\beta s}{\beta^2 s^2}ds = \int_t^\infty \frac{1+\cos 2\beta s}{2\beta s^{22}}ds$$

$$= \int_t^\infty \frac{1}{2\beta^2 s^2}ds + \int \frac{\cos 2\beta s}{2\beta^2 s^2}ds$$

$$= \frac{1}{2\beta^2 t} + 0\left(\frac{1}{t^2}\right). \quad (20)$$

Now applying condition (16) using (20) and (18), gives the left hand side as

$$\int_t^\infty \left(\frac{1}{2\beta^2 s}\right)^2 ds = \frac{1}{4\beta^4}\int_t^\infty \frac{1}{s^2}ds = \frac{1}{4\beta^4 s}\Bigg|_t^\infty$$

$$= \frac{1}{4\beta^4 t}. \tag{21}$$

Now the right hand side of equation (16)

$$= \frac{1}{4\times 2\beta^2 t} = \frac{1}{8\beta^2 t}. \tag{22}$$

Thus comparing (21) and (22) according to (16)

$$\frac{1}{4\beta^4 t} > \frac{1}{8\beta^2 t}. \tag{23}$$

Rearranging the inequalities in (23) we have

$$\frac{1}{\beta^2} > \frac{1}{2} \Rightarrow \frac{1}{\beta} > \pm\sqrt{\frac{1}{2}}$$

$$\left|\frac{1}{\beta}\right| > \frac{1}{\sqrt{2}}. \tag{24}$$

Condition (16) implies oscillation of (12) if

$$\left|\frac{1}{\beta}\right| > \frac{1}{\sqrt{2}}.$$

Result using the Integral Technique

Now consider the case where $q(t)$ in (12) is given the form

$$q(t) = \frac{\sin\beta t}{t^\lambda}, \text{ for } \beta\neq 0 \ \lambda>1 \tag{25}$$

Case 1:

Consider the case where *λ=2* Equation (25) can be written as

$$q(t) = \frac{\sin\beta t}{t^2} \tag{26}$$

Thus using equation (3),

$$Q(t) = \int_t^\infty q(s)ds = \int_t^\infty \frac{\sin\beta s}{s^2}ds$$

$$= -\frac{\cos\beta s}{\beta s^2}\Bigg|_t^\infty - 2\int_t^\infty \frac{\cos\beta s}{\beta s^3}ds$$

$$= \frac{\cos\beta t}{\beta t^2} + 0\left(\frac{1}{t^3}\right) \tag{27}$$

and using equation (4),

$$P_Q(s,t) = \exp\left(2\int_t^s Q(\tau)d\tau\right) = 1 + 2\int_t^s Q(\tau)d\tau + \ldots$$

$$= 1 + 2\int_t^s \left\{\frac{\cos\beta\tau}{\beta\tau^2} + 0\left(\frac{1}{\tau^3}\right)\right\}d\tau + \ldots \tag{28}$$

$$= 1 + 0\left(\frac{1}{t^3}\right) \tag{29}$$

then from equation(27) we have

$$Q^2(t) = \frac{\cos^2\beta t}{\beta^2 t^4}. \tag{30}$$

Substituting (30) and (29) into (14) gives

$$\overline{Q}(t) = \int_t^\infty Q^2(s)P_Q(s,t)ds = \int_t^\infty \frac{\cos^2\beta s}{\beta^2 s^4}ds = \int_t^\infty \frac{1+\cos 2\beta s}{2\beta^2 s^4}ds$$

$$= \int_t^\infty \frac{1}{2\beta^2 s^4}ds + \int_t^\infty \frac{\cos 2\beta s}{2\beta^2 s^4}ds = -\frac{1}{6\beta^2 s^3}\Bigg|_t^\infty + 0\left(\frac{1}{t^4}\right)$$

$$= \frac{1}{6\beta^2 t^3} + 0\left(\frac{1}{t^4}\right). \tag{31}$$

Now applying condition (16) using (31) and (29), gives the left hand side as

$$\int_t^\infty \left(\frac{1}{6\beta^2 s^3}\right)^2 ds = \frac{1}{36\beta^4}\int_t^\infty \frac{1}{s^6}ds = -\frac{1}{36\times 5\beta^4 s^5}\Bigg|_t^\infty$$

$$= \frac{1}{180\beta^4 t^5} \tag{32}$$

and right hand of equation (16) as

$$\frac{1}{4\times 6\beta^2 t^3} = \frac{1}{24\beta^2 t^3}. \tag{33}$$

Thus comparing (32) and (33) according to (16), we have

$$\frac{1}{180\beta^4 t^5} > \frac{1}{24\beta^2 t^3}. \tag{34}$$

Now rearranging the inequality in (34), we have

$$\frac{24\beta^2 t^3}{180\beta^4 t^5} > 1 \ \Rightarrow \frac{2}{15\beta^2 t^2} > 1$$

$$\Rightarrow \frac{1}{\beta t} > \pm\sqrt{\frac{15}{2}}$$

$$\Rightarrow \left|\frac{1}{\beta t}\right| > \sqrt{\frac{15}{2}} \Rightarrow \left|\frac{1}{\beta}\right|\left|\frac{1}{t}\right| > \frac{1}{\sqrt{2}}\sqrt{15}$$

$$\left|\frac{1}{\beta}\right| > \frac{1}{\sqrt{2}} \text{ and } \left|\frac{1}{t}\right| > \sqrt{15}. \quad (35)$$

We observe from (16) that oscillation of (26) depend on β and t thus condition (1) implies oscillation if

$$\left|\frac{1}{\beta}\right| > \frac{1}{\sqrt{2}} \quad \text{and} \quad \left|\frac{1}{t}\right| > \sqrt{15} \quad (36)$$

Case 2

In this case, condition (16) is applied on $q(t)$, which is given in (25).Using (3), (5) and (7) we have

$$Q(t) = \int_t^\infty q(s)ds = \int_t^\infty \frac{\sin\beta s}{s^\lambda} ds$$

$$-\left.\frac{\cos\beta s}{\beta s^\lambda}\right|_t^\infty - 2\int_t^\infty \frac{\cos\beta s}{\beta s^{\lambda+1}} ds$$

$$= \frac{\cos\beta t}{\beta t^\lambda} + 0\left(\frac{1}{t^{\lambda+1}}\right) \quad (37)$$

and

$$P(s,t) = \exp\left(2\int_t^s Q(\tau)d\tau\right) = 1 + 2\int_t^s Q(\tau)d\tau + \ldots$$

$$= 1 + 2\int_t^s \left\{\frac{Cos\beta\tau}{\beta\tau^\lambda} + 0\left(\frac{1}{\tau^{\lambda+1}}\right)\right\} d\tau + \ldots$$

$$= 1 + 0\left(\frac{1}{\tau^{\lambda+1}}\right). \quad (38)$$

Then from equation (37), we have

$$Q^2(t) = \frac{\cos^2 \beta t}{\beta^2 t^{2\lambda}}. \quad (39)$$

Substituting (39), and (38) into (14), gives

$$\overline{Q}(t) = \int_t^\infty Q^2(s)P_Q(s,t)ds = \int_t^\infty \frac{\cos^2 \beta s}{\beta^2 s^{2\lambda}} ds = \int_t^\infty \frac{1+\cos 2\beta s}{2\beta^2 s^{2\lambda}}$$

$$= \int_t^\infty \frac{1}{2\beta^2 s^{2\lambda}} ds + \int_t^\infty \frac{\cos 2\beta s}{2\beta^2 s^{2\lambda}} ds = -\left.\frac{1}{2(2\lambda-1)\beta^2 s^{2\lambda-1}}\right|_t^\infty + 0\left(\frac{1}{t^{2\lambda}}\right)$$

$$= \frac{1}{2(2\lambda-1)\beta^2 t^{(2\lambda-1)}} + 0\left(\frac{1}{t^{2\lambda}}\right). \quad (40)$$

Considering oscillation condition (16), gives the left hand side as

$$\int_t^\infty \left(\frac{1}{(4\lambda-2)\beta^2 s^{(2\lambda-1)}}\right)^2 ds = \frac{1}{(4\lambda-2)^2\beta^4}\int_t^\infty \frac{1}{s^{(4\lambda-2)}} ds = -\left.\frac{1}{(4\lambda-2)^2(4\lambda-3)\beta^4 s^{(4\lambda-3)}}\right|_t^\infty$$

$$= \frac{1}{(4\lambda-2)^2(4\lambda-3)\beta^4 t^{(4\lambda-3)}}. \quad (40)$$

The right hand side is

$$\frac{1}{4(4\lambda-2)\beta^2 t^{(2\lambda-1)}}. \quad (41)$$

Thus comparing (40), and (41), according to (16) we have the inequality

$$\frac{1}{(4\lambda-2)^2(4\lambda-3)\beta^4 t^{(4\lambda-3)}} > \frac{1}{4(4\lambda-2)\beta^2 t^{(2\lambda-1)}}. \quad (42)$$

Now rearranging the inequality (42) we have

$$\frac{4(4\lambda-2)\beta^2 t^{(2\lambda-1)}}{(4\lambda-2)^2(4\lambda-3)\beta^4 t^{(4\lambda-3)}} > 1$$

$$\Rightarrow \frac{4}{(4\lambda-2)(4\lambda-3)\beta^2 t^{2(\lambda-1)}} > 1$$

$$\frac{2}{(2\lambda-1)(4\lambda-3)(\beta t^{\lambda-1)})^2} > 1 \quad (43)$$

$$\left|\frac{1}{t^{(\lambda-1)}\beta}\right| > \sqrt{\frac{(2\lambda-1)(4\lambda-3)}{2}}. \quad (45)$$

The inequality (45) implies that

$$\left|\frac{1}{\beta}\right| > \frac{1}{\sqrt{2}} \quad \text{and}$$

$$\left|\frac{1}{t^{(\lambda-1)}}\right| > \sqrt{(2\lambda-1)(4\lambda-3)}$$

This shows that using $q(t) = \frac{\sin \beta t}{t^{\lambda}}$, for condition (16) implies that it is oscillatory if

$$\left|\frac{1}{\beta}\right| > \frac{1}{\sqrt{2}} \quad \text{and}$$

$$\left|\frac{1}{t^{(\lambda-1)}}\right| > \sqrt{(2\lambda-1)(4\lambda-3)}. \qquad (46)$$

DISCUSSION

The result obtained in (46) is the generalized form of Wong's criteria for oscillation of second order linear differential equations. It is observed that when $\lambda = 1$ in condition (46), we have Wong's result that is $\left|\frac{1}{\beta}\right| > \frac{1}{\sqrt{2}}$. Similarly, when $\lambda = 2$ in(46) the result obtained is illustrated in case one, which indicates that it is oscillatory when $\left|\frac{1}{\beta}\right| > \frac{1}{\sqrt{2}}$ and $\left|\frac{1}{t}\right| > \sqrt{15}$.

[1] technique identify the criteria for oscillation of differential equation been investigated.

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