CHAPTER 1

VECTORS

1.1.0 Vector and scalars

Vector methods have become standard tolls for the physicists. In this chapter we discuss the properties of the vectors and vectors fields that occur in classical physics. We will do so in a way, and in a notation, that leads to the formation of abstract linear vector spaces.

A physical quantity that is completely specified, in appropriate units, by a single number (called its magnitude) such as volume, mass, and temperature is called a scalar. Scalar quantities are treated as ordinary real numbers. They obey all the regular rules of algebraic addition, subtraction, multiplication, division, and so on.

There are also physical quantities which require a magnitude and a direction for their complete specification. These are called vector if their combination with each other is commutative (that is the order of addition may be changed without affecting the result). Thus not all quantities possessing magnitude and direction are vectors. Angular displacement, for example, may be characterized by magnitude and direction but is not vector, for the addition of two or more angular displacements is not, in general, commutative.

Generally, we shall denote vector by boldface letters (such as A) and use ordinary italic letters (such as A) for their magnitudes; in writing, vectors are usually represented by a letter with an arrow an above it such as$\vec{A}$. A given vector A (or$\vec{A}$.) can be written as

$$A = A\vec{A}. , \qquad (1.1)$$

Where A is the magnitude of vector Aand so it has unit and dimension, and $\hat{A}$ is a dimensionless unit vector with unity magnitude having the direction of A. thus$\hat{\vec{A}}. = A/A$.

A vector quantity maybe represented graphically by an arrow-tipped line segment. The length of the arrow represents the magnitude of the vector, and the direction of the arrow is that of the vector, as show in Fig 1.1. Alternatively, a vector can be specified by its components (projections along the coordinate axes) and the unit vectors along the coordinate axes (Fig 1.2).

$$A = A_1\hat{e}_1 + A_2\hat{e}_2 + A\hat{e}_3 = \sum_{i=1}^{3} A_1\hat{e}_1 \qquad (1.2a)$$

Where $\hat{e}_i (i = 1,2,3)$ are unit vector along the rectangular axes x_i $(x_1 = x, x_2 = y, x_3 = z)$; they are normally written as $\hat{\imath}, \hat{\jmath}, \hat{k}$ in general physics textbooks. The component triplet (A_1, A_2, A_3) is also often used as an alternate designation for vector A

$$A = A_1 e_1 \qquad (1.2b)$$

This algebraic notation of a vector can be extended (or generalized) to spaces of dimension greater than three, where an ordered n −tuple of real numbers, $(A_1, A_2, \cdots, A_n)$, represents a vector. Even though we cannot construct physical vectors for $n > 3$, we can retain the geometrical language for these n −dimensional generalizations.

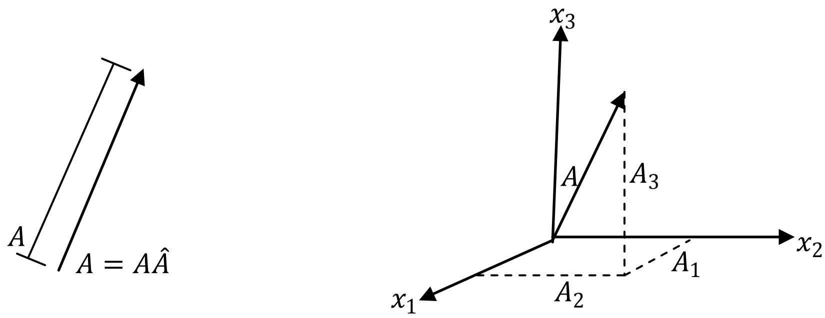


Figure 1.2.A vector A in Cartesian coordinates.

1.1.1 Direction angles and direction cosines

We express the unit vector $\hat{A}$ in terms of the unit coordinate vectors$\hat{e}_1$. From Eq (1.2), $A = A_1\hat{e}_1 + A_2\hat{e}_2 + A\hat{e}_3$, we have

$$A = A\left(\frac{A_1}{A}\hat{e}_1 + \frac{A_2}{A}\hat{e}_2 + \frac{A_3}{A}\hat{e}_3\right) = A\hat{A}$$

Now $\frac{A_1}{A} = \cos\alpha$, $\frac{A_2}{A} cos\,\beta, and \frac{A_1}{A} = \cos\gamma$ are the direction cosines of the vector A, and a, β, and γ are the direction angles (Fig. 1.3). Thus we can write

$$A = A(\cos\alpha\hat{e}_1 + \cos\beta\hat{e}_2 + \cos\gamma\hat{e}_3) = A\hat{A};$$

It follows that

$$\hat{A} = (\cos a\hat{e}_1 + \cos\beta\hat{e}_2 + \cos\gamma\hat{e}_3) = (\cos a, \cos\beta, \cos\gamma). \qquad (1.3a)$$

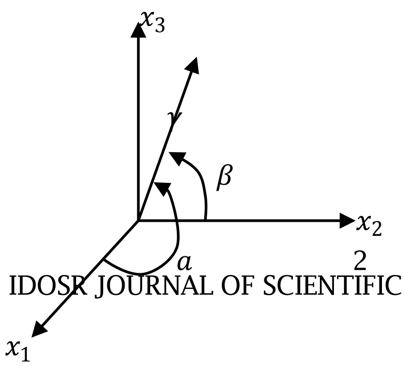


Figure 1.3 Direction angles of vector

Two vectors, say **A** and **B,** are equal if, and only if, their respective components are equal:

$A = B or (A_1, A_2, A_3)\ (B_1, B_2, B_3)$ (1.3b)

Is equivalent to the three equations?

$$A_1 = B_1, A_2 = B_2, A_3 = B_3.$$

Geometrically, equal vector are parallel and have the same length, but do notnecessarily have the same position.

1.1.2 Vector addition

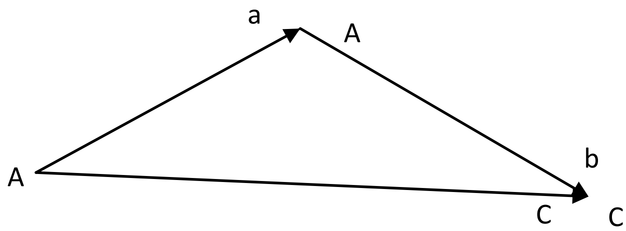
The addition of two vectors is defined by the equation

$A = B = (A_1, A_2, A_3) + (B_1, B_2, B_3) + (A_1 + B_1, A_2 + B_2, A_3 + B_3)$ (1.4)

That is, the sum of two vectors is a vector whose components are sums of the components of the two given vectors.

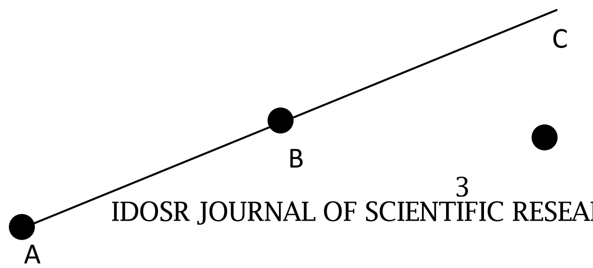
We can add two non-parallel vectors by graphical method as shown in Fig. 1.5. To add vector B vector A, shift B parallel to itself until is tail is at the head of A. the vectors sum A+B is a vector C drawn from the tail of A to the head of **B.** the order in which the vector are added does not affect the result.

The addition of two vectors is defined by the triangle law:



Let AB and BC represent the vectors a and b. the addition of vectors a and b is written a+b and is defined by a+b=AC, thus AB+BC=AC where AC represents the vector c.

If the points A, B, and C are collinear, the law of vector addition still requires that AB+BC= AC. Although the triangle ABC is no more, the magnitudes AB+BC=AC



If C were coincident with A in the first case then AC=0 and we define

AC=0

Where 0 is the zero or rule vector If A and C are coincident we may also

Write AB+BC-0

And hence AB+BA=0

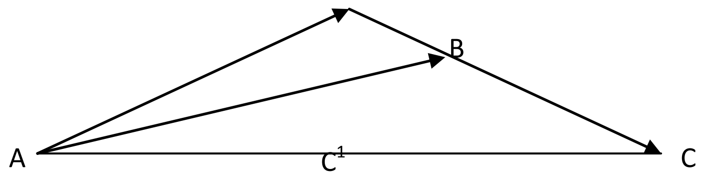
AB=-BA

Which defines our use of the minus sign AB and BA are vectors with equal magnitude but opposite directions.

By definition to subtract U from V, we add V and (-U). so

V-U=v+ (-u)

We have the triangle law for the addition of vectors, and we know that –u and u are vectors of equal magnitude in opposite directions.



If is the mid- point of CC^1

BC^1= - BC

AC^1=AB+BC^1

AC^1=AB-BC

1.1.3 Multiplication by a scalar

If c is scalar then

$cA = (cA_1, cA_2, cA_3)$. (1.5)

Geometrically, the vector cA is parallel to A and c times the length of A. when $c = -1$, the vector$-A$ is one whose direction is the reverse of that of A, but both

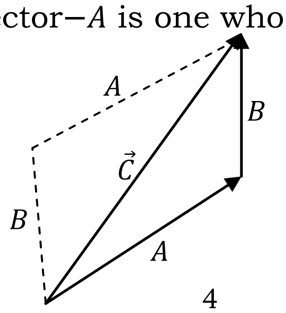


Figure 1.4 Addition of two vectors.

Have the length. Thus, subtraction of vector B from vector A is equivalent to adding$-B$ to A;

$A - B = A + (-B)$. (1.6)

We see that vector addition has the following properties:

a. $A + B = B + A$(commutativity);
b. $(A + B) + C = A + (B + C)$(associativity);
c. $A + 0 = 0 + A = A$;
d. $A + (-A) = 0$.

We now turn to vector multiplication. Note that division by a vector is not defined: expressions such as k/A or B/A are meaningless.

There are several ways of multiplying two vectors. Each of which has a specialmeaning tow types are defined.

1.1.4 LAWS OF VECTOR ALGEBRA

Vector algebra involve the following rules

1. A+B=B+A commutative law
2. A+(B+C0=(A+B) +C= Associative law
3. mA=Am commutative
4. (m+n) A=mA+ n A distributive law m and n are scalar qualities
5. (m +n)A= mA+ n A distributive law
6. m (A +B)=mA+mB)

1.1.5 DIRECTION COSINES

In the previous diagram, the anglesα , and γ are between op and j and k respectively; cos α, cos β and cos γ are called the direction cosines of op

$a = op \cos \alpha$

$b = op \cos \beta$

$c = op \cos \gamma$

Already we have seen that

$0p^2 = a^2 + b^2 + c^2$

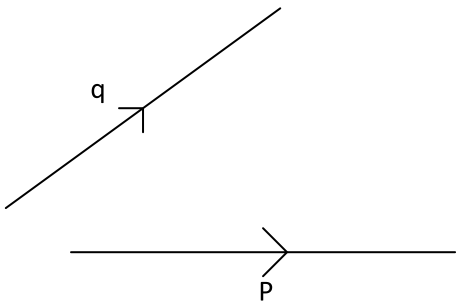
$\therefore 0p^2 = 0p^2 \cos^2\alpha + 0p^2 \cos^2\beta + 0p^2 \cos^2\gamma$

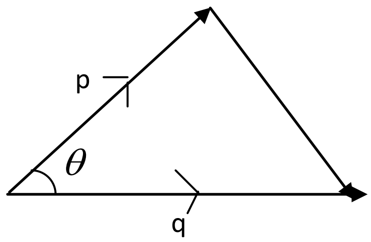
$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

We may now write

$$(a)\gamma_A reloB = \gamma_A - \gamma_B = (3i + 4j\text{-}5k) - (-2i - j + k)$$

1.2 ANGLE BETWEEN ANY TWO VECTORS





We use cosine formula

$$\| q - p \| 2 = q^2 + p^2 - 2pq \cos \theta$$

let $q=q(l_1 i + m_1 j + n_1 k)$ *and* $p = p\ (l_1 i + m_1 j + n_1 k)$

where l,m, n, and l_1, m_1, and n_1, are

Direction cosines of q and p respectively

Therefore

$$= \frac{\cos\ \theta = q^2 + p^2 - (ql - pl_1)^2 - (qm - pm_1)^2 - (qn - pn_1)^2}{2qp}$$

$$= \frac{q^2 (l^2 - m^2 - n^2) + p[l^2 - l_1^2 - m_1^2 - n_1^2 + 2pq\,(ll_1 + mm_1 + nn_1)}{2pq}$$

$= ll_1 + mm_1 + nn_1$

since $l^2 + m^2 + m^2 = 1$ *and*

$l_1^2 + m_1^2 + n_1^2 = 1$

As we have already indicated in discussion of direction

Cosines

Example 1

Given that $A = -i + 2j - 2k$ find

a. The magnitude of A
b. The unit vector in the direction of A
c. The direction cosine of A
d. The angle between A and $B = 3i + 4j - 12k$
e. The angle between 3A-B and A+B
f.

SOLUTION

$$|A| = \left[(-1)^2 + (2)^2 + ((-2)^2)\right]^{2} = \sqrt{9} = 3$$

$$(b)\ e_A = \frac{A}{|A|} = \frac{-i + 2j - 2k}{3} = \frac{-1}{3}i + \frac{2}{3}j - \frac{2}{3}k$$

(c) the direction cosine of A are; $-\frac{1}{2,}\ \frac{2}{3}\ and\ -\frac{2}{3}$

$B =$ 3i + 4j - 12k

$$(d)\ e_B = \frac{B}{|B|} = \frac{3i + 4j - 12k}{\sqrt{3^2 + 4^2 + (-12)^2}} = \frac{3i + 4j - 12k}{13}$$

Therefore the direction cosines of b are $\frac{3}{13,}\ \frac{4}{13}$ and $-\frac{12}{13}$

If the angle between the two vectors A and B is θ, we can use direction cosine to obtain the value of θ In direction cosine of

A, e_A $1 = -\frac{1}{3,}$ m=$\frac{2}{3,}$ n=$-\frac{2}{3}$ and in that of

B, L, $= -\frac{3}{13,}$ m=$\frac{4}{13,}$ n=$-\frac{12}{13}$

$\therefore COS\ \theta$ = ll, + mm, + nn,

$$= -\frac{1}{3}\left(\frac{3}{13}\right) + \left(\frac{2}{3}\right)\left(\frac{4}{13}\right) + \left(-\frac{2}{3}\right)\left(-\frac{12}{13}\right) = \frac{29}{39}$$

$$\theta\ \cos^{-1}\left(\frac{29}{39}\right) = 47^0$$

(e) Define

A^1 andB^1 by

$A^1 = $ 3A-B=(-3i + bj - 6k) − (3i + 4j - 12k) = -6i + 2j + 6k

$b^1 = $ A + B = (- i + 2j -2) + (+ 3i + 4j -12k) = 2i + 6j 14k

$$eA^1 = \frac{-6i + 2j + 6k}{\left[(-6)^2 + (2)^2 + (6)^2\right]^{1/2}} = \frac{-6i + 2j + 6k}{\sqrt{76}}$$

$$= \frac{-3I + J + 3K}{\sqrt{19}}$$

$$eB^1 = \frac{2i + 6j - 14k}{\left[(2)^2 + (6)^2 + (-14)^2\right]^{1/2}} = \frac{2i + 6j - 14k}{\sqrt{236}}$$

$$= \frac{i + 3j - 7k}{\sqrt{59}}$$

If the angle between A^1 and $B^1$is θ, then

$$\cos\ \theta = \left(\frac{-3}{\sqrt{19}}\right)\left(\frac{1}{\sqrt{59}}\right) + \left(\frac{3}{\sqrt{59}}\right) + \left(\frac{3}{\sqrt{19}}\right)\left(\frac{-7}{\sqrt{59}}\right)$$

$$= \frac{-21}{\left(\sqrt{19}\right)\left(\sqrt{59}\right)} = -0.62722$$

$$\phi = \cos^{-1}(-0.62722) = 129^\circ$$

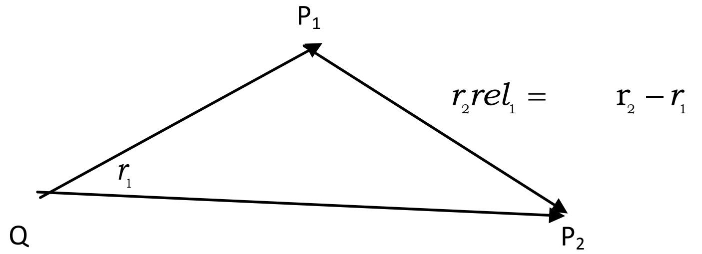
1.3 RELATIVE VECTORS

This concept involves the application of a vector method to different types of problems such as

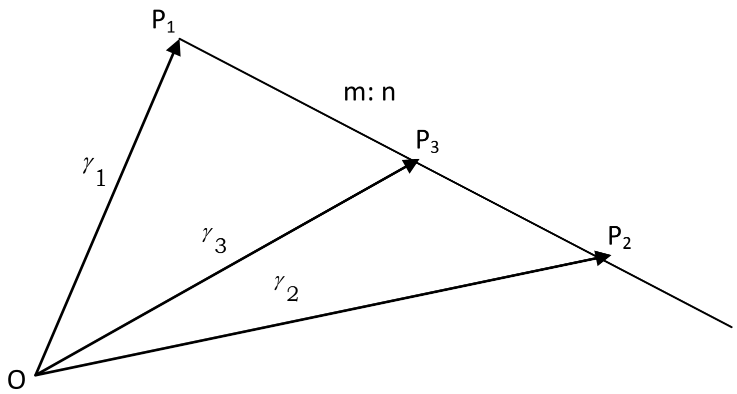
(a) Position vectors.

The position vector defines the position of one point relative to another point

$r_2 rel\ 1$



Which reads; the position vector of p_2 relative to $p_1$is equal to the position vector of p_2 minus the position vector of p_1 co linearitypoints



$$p_1 p_1 = r_2 - r_1 \text{ and } p_3 p_2 = r_2 - r_3$$
$$\text{therefore } n(r_3 - r_1) = m(r_2 - r_3)$$
$$r_3 = \frac{n\, r_1 + m\, r_2}{n+m}$$

If we are considering external division m or n will be negative. The position vectors specified in the diagram relative to, o, r_1 and r_2 defines the division of a given straight line in a given ratio say m: n and provides a good test for co linearity of three points.

The condition for co linearity may be written
$mr_1 + mr_2 + lr_3 = 0$ *where* n + m + l = 0
if r_1, r_2 *and* r_3

Are specified in terms of equating components of vectors

$$na_1 + mb_1 = (n+m)c_1$$
$$na_2 + mb_2 = (n+m)c_2$$
$$na_3 + mb_3 = (n+m)c_3$$

These equations must each give the same relationship between n and m if r_1, r_2,

Example 2

Particle A is at position 3i + 4j-5k moving with a velocity -2i+3j + 3j +6k and acceleration $i\alpha j - 2k$ particle B is at position -2i-j+k moving with a velocity i-2j + 2k and acceleration –i-2j +k

Find (a) the position of a relative to B

(b)Velocity of B relative to A

(a) Acceleration of B relative to A

Solution

$(a)\gamma_A relB = \gamma_A - \gamma_B = (3i\ +\ 4j\text{-}5k) - (-2i - j + k) = 5i\ +\ 5j\ -\ 6k$ in meters^{-2}

$v_B - v_A =\ v_B rel.B$

b. $= (i - 2j + 2k) - (-2i + 3j + 6k)$

$= 3i - 5j - 4k$ in meters^{-2}

$a_{BrelA} = a_B - a_A = (\text{-}i\text{-}2j\text{+ }k)\text{=}(i\text{+}j\text{–}2k)$

$= -2i$ -3j +3k in meter s^{-2}

Example 3

The position vectors of A and B are $\gamma_A = 4i\ +\ 6j\ -\ 8k$ and $\gamma_B = 3i\ \text{-}2j\ +\ k$

(a) Find the position vector of C, which divides AB in the ratio 2:-1
(b) Find the position vector X, which divides BA in the ratio 2:1
(c) Determine whether point $y, \gamma_z =\ 6i + 22j\ \text{-}26k$ is collinear with A and b.

Solution

$$\gamma_C = \frac{n\gamma_A + m\gamma_B}{n+m} \text{ where min = 2:1}$$

$$= \text{-}1\ \frac{(4i\ +\ bj\text{-}k) + 2(3i = 2j + k)}{-1+2}$$

a. $=\ 2i\text{-}10j\ +\ 10k$

$$b\ \gamma_x \frac{n\gamma_A + m\gamma_B}{n+m} = \frac{(3i - 2j + k) + 2(4i + bj - 8k)}{1+2}$$

$$= \frac{11i\ +\ bj\ \text{-}15k}{3}$$

We use the condition for γ, A and B mast be collinear. to be collinear with A and B"

$n(4i\ +\ 6j\ -\ 8k) + m(3i\ -\ 2j\ +\ k) = (m+n)(6i + 22j - 26k)4n + 3m = 6(m+n) \rightarrow 2n + 3m = 0$

$6n + 2m = 22(m+n)\ \ 16n + 24m = 0$

$18n\text{+}m\text{=-}26(m\text{+}n)18n + 27m = 0$

This gives the same relationship between M and n in the equations we can also use

$\gamma_A = \gamma_A - \gamma_y = (4i + 6j - 8k) - (6i + 22j - 26k)$

$= -2\ (i + 8j - 9k)$

$yB\ \gamma_B - r_y = (3i + 2j + k) - (6i + 22j - 26k)$ Which is an indication

$= -3\ (i + 8j - 9k)$

thus 3yA = 2ybB

that γ, A and B mast be collinear.

1.3.0 Scalar product of vectors

The scalar product of A and B are written A.B = AB e_n

That is if A = $a_1i + a_2j + a_3k$ and B = $b_1i + b_2j + b_3k$

The properties of the scalar product

(a) Commutative law; A.B = B.A

i.e A.B = AB $\cos\theta$ = BA $\cos\theta$ =B.A

(b) Distributive law; A. (mB) = m (A.B) which means that

m(A .B) = mAB $\cos\theta$

A. (mB) = AmB $\cos\theta$

A. mB) = m (A.B)

Hence A, (mB) = m.(A.B)

(c) A.(C+D) = A.C A.D

We note that if

i. $\theta = 0$, the vectors A ad B have the same direction A.B = AB $\cos\theta$ = AB

ii. $\theta = \pi$ means that A and B is directly opposite B and A.B = AB $\cos\pi$ = - AB

iii. $\theta = \frac{\pi}{2}$ Means that A and B are perpendicular and A.B = ABcosθ =0

iv. i.i= j.j = k.k=1 and conversely k.i=0, j.i = k.j = 0

Example 4

If forces fi = 3i + 5j + 6k and f_2 =2i + 3j – 4k newtons are displaced d = 3i 02j + 4k meters find the work done

Solution

Total force F_t = $f_1 + f_2$ = 5i + 8j + 2k work done is F_t x d = (5i + 8j + 2k). (3i – 2j + 4k)

= 15 + 16 + 8 = 7Nm

Example 5

The angle between A and B is $Cos^{-1}\left(\frac{4}{21}\right)$. Find P given that A = 6i + 3j-2kj B=2i + pj -4k

Solution

A.B = (6i + 3j -2K). (-2i + Pj -4K)

4 **= -**12 + 3p + 8; 4= -4 + 3p

$$p = \frac{8}{3}$$

1.3.1 Vector product

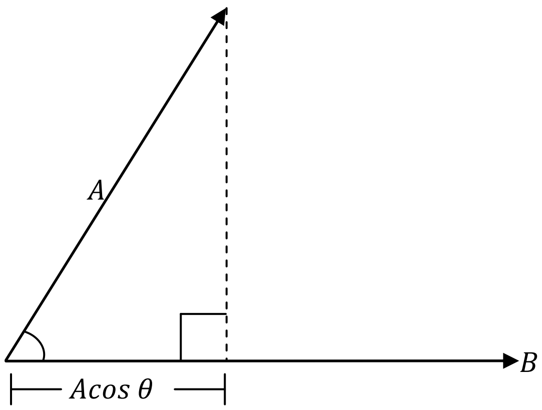


Fig.1.5 Shows vector A and B

We can get a simple geometric interpretation of the dot product from an inspection of Fig. 1.5.

$(B\cos\theta)a$ = Projection of **B** onto **A** multiplied by the magnitude of A,

$(A\cos\theta)B$ =Projection A onto B multiplied by the magnitude of B.

If only the components of A and B

Are known, then it would not be practical to calculate $A.B$ for definition (1.4). But, in this case, we calculate $A.B$ in terms of the components:

$$A.B = (A_1\hat{e}_1 + A_2\hat{e}_2 + A\hat{e}_3).(B_1\hat{e}_1 + B_2\hat{e}_2 + B\hat{e}_3); \qquad (1.7)$$

The right hand side has nine terms, all involving the product$\hat{e}_i.\hat{e}_j$. Fortunately, the angle between each pair of unit vectors is 90^0, and from (1.4) and (1.6) we fine that

$$\hat{e}_i.\hat{e}_j = \delta_{ij}, \qquad i,j = 1,2,3 \qquad (1.8)$$

Where δ_{ij} is the Kronecker delta symbol

$$\delta_{ij} \begin{cases} 0, if i \neq j, \\ 1, if i = j. \end{cases} \qquad (1.9)$$

After we use (1.8) to simplify the resulting nine terms on the right-side of (7), we obtain

$$A.B = A_1, B_1 + A_2, B_2 + A_3, B_3 = \sum_{i=1}^{3} A_i B_i.$$

The law of cosines for plane triangles can be sassily proved with the application of the scalar produce; refer to Fig 1.1 where C is the resultant vector of A and B. taking the dot product of C with itself, we obtain.

$$C^2 = C.C = (A +).(A + B)$$

$$= A^2 + B^2 + 2A.B = A^2 + B^2 + 2Ab\cos\theta,$$

Which is the law of cosines

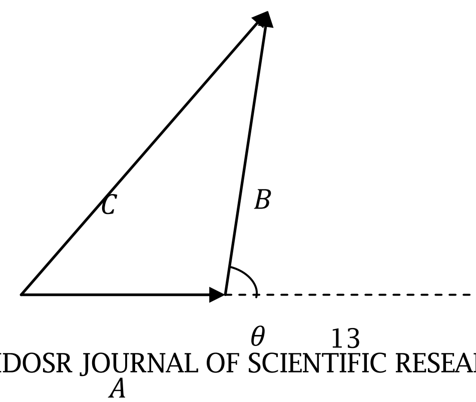


Figure 1.6 Law of cosines

A simple application of the scalar product in physics is the work W done by a constant force F: $W = F.r$ where r is the displacement vector of the object moved by F.

1.3.2 The vector (cross or outer) product

The vector product of two vectors A and B form two sides of a parallelogram. We define C to be perpendicular to the plane of this parallelogram with its magnitude equal to the area of the parallelogram. And we choose the direction of C along the thumb of the right hand when the fingers rotate from$A$ to B (angle of rotation less than 180^0).

$$C = A \times B = AB \sin\theta \hat{e}_c \; (0 \leq \theta \leq \pi) \qquad (1.12)$$

From the definition of the vector product and following the right hand rule, we can see immediately that

$$A \times B = -B \times A \qquad (1.13)$$

Hence the vector product is not commutative. If A and B are parallel, then it follows from $Eq.$(1.12) that

$$A \times B = 0 \qquad (1.14a)$$

In particular

$$A \times A = 0 \qquad (1.14b)$$

In vector components, we have

$$A \times B = (A_1\hat{e}_1 + A_2\hat{e}_2 + A\hat{e}_3) \times (B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3); \qquad (1.15)$$

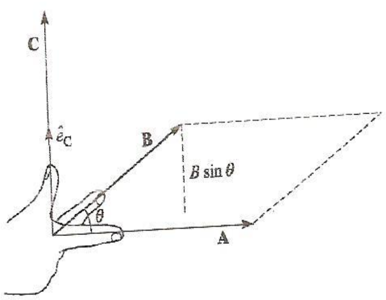


Figure 1.7 The right hand rule for vector product

Using the following relations

$$\hat{e}_i \times \hat{e}_i = 0, i = 1,2,3,$$

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3, \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \hat{e}_3 \times \hat{e}_1 = \hat{e}_2, \qquad (1.16a)$$

$Eq.$ (1.15)becomes

$$A \times B = (A_2B_3 - A_3B_2)\hat{e}_1 + (A_3B_1 - A_1B_3)\hat{e}_2 + (A_1B_2 - A_2B_1)\hat{e}_3 \qquad (1.16b)$$

This can be written as an easily remembered determinant of third order:

$$A \times B = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \qquad (1.17a)$$

The expansion of determinant of third order can be obtained by diagonal multiplication by repeating on the right the first two columns of the determinant and adding the signed products of the elements on the various diagonals in the resulting array:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{matrix} \qquad (1.17b)$$

The non-commutatively of the vector product of two vector now appears as a consequence of the fact that interchanging two rows of determinant changes its sign, and the vanishing of the vector product of two vectors in the same direction appears as a consequence of the fact that a determinant vanishes if one of its rows is a multiple of another.

The determinant is a basic tool used in physics and engineering. The reader is assumed to be familiar with this subject.

The vector resulting from the vector product of two vector is called an axial vector wile ordinary vectors are sometimes called polar vectors. Thus, in $Eq.(1.11)$, C is a pseudo vector, while A and B are axial vectors. One an inversion of coordinates, polar vectors change sign but an axial vector does not change sign.

A simple application of the vector production in physics is the torque r of a force F about a point$O : r = F \times r$, where r is the vector from O to the intial point of the force F(Fig. 1.8).

We can write the nine equations implied by Eq (1.16) in terms of permutation symbols ε_{ijk}:

$$\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk}\,\hat{e}_{k,} \tag{1.18}$$

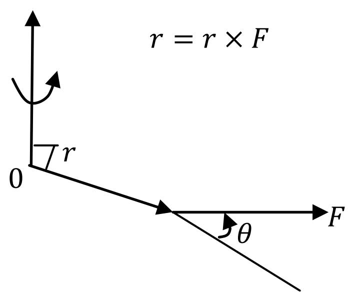


Figure 1.8 the torque of a force about a point 0

Where ε_{ijk} is defined by

$$\varepsilon_{ijk}\begin{cases} +1\ if(i,j,k) \text{ is an even permutation of}(1,2,3), \\ -0\ if(i,j,k)\text{is an odd permutation of}(1,2,3), \\ \quad 0\ \ otherwise\text{(for example, if 2 or more indices are equal)} \end{cases} \tag{1.19}$$

It follows immediately that

$$\varepsilon_{ijk} = \varepsilon_{ijk} = \varepsilon_{ijk} = -\varepsilon_{ijk} = -\varepsilon_{ijk} = -\varepsilon_{ijk}\,.$$

There is a very useful identity relating the ε_{ijk} and the Kronecker delta symbol:

$$\sum_{k=1}^{3} \epsilon_{mn\,k}\, \varepsilon_{ijk} = \delta_{mi}\,\delta_{nj} - \delta_{mj}\,\delta_{ni}, \qquad (1.20)a$$

$$\sum_{j,k} \varepsilon_{mjk}\ \varepsilon_{njk} = 2\delta_{m}, \sum_{i,j,k} \varepsilon_{ijk}^{2} = 6 \qquad (1.20b)$$

Using permutation symbols, we can now write the vector product $A \times B$ as

$$A \times B = \left(\sum_{i=1}^{3} A_i \hat{e}_i\right) \times \left(\sum_{j=1}^{3} B_j \hat{e}_j\right) = \sum_{i,j}^{3} A_i B_j \left(\hat{e}_i \times \hat{e}_j\right) = \sum_{i,j,k}^{3} (A_i B_j\, \varepsilon_{ijk})\hat{e}_k.$$

Thus the kth component of $A \times B$ is

$(A \times B)_1 = \sum_{i,j} A_i B_j\, \varepsilon_{ijk} = \varepsilon_{ij\,k}$ (1.21)

If $k = 1$, we obtain the usual geometrical result:

$(A \times B)_1 \sum_{i,j} \varepsilon_{ijk}\, A_i B_j = \varepsilon_{123} A_2 B_3 + \varepsilon_{123} A_3 B_2 = A_2 B_3 - A_3 B_2$. (1.22)

The vector product of any two vector a and b is written as Ax B = AB Sin θe_n where $\theta =$ angle between the vector a and B while e_n is a unit vector perpendicular to the plane containing a and B

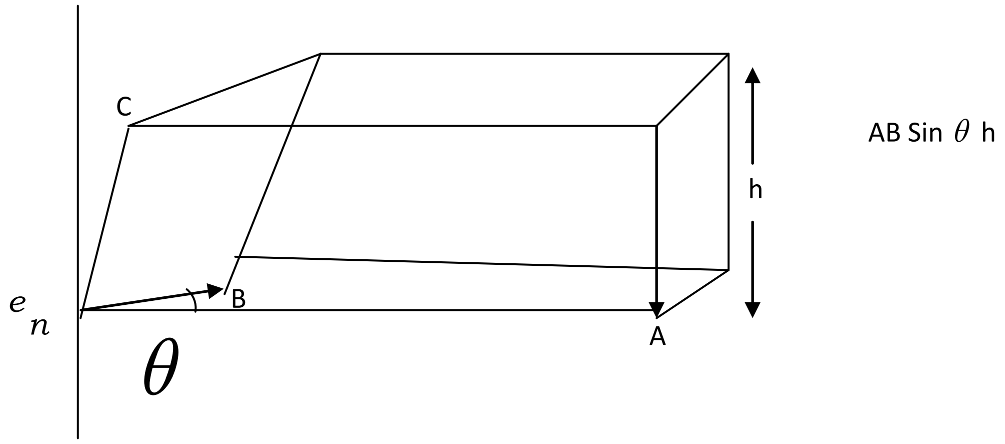
properties of vector product B x A = - A B Sin θe_n

Meaning that b x A -= (a X B) which implies that vector product is not commutative vector product is applied in finding the arece of a parallelogram whose sides are formed by vectors A and B. It can also be used in finding the area of triangle which has two of its sides formed by the vectors A and B and is given by

Area =

It is especially employed in determination of a parallelepiped and scalar triple product which of vital use in cry stall graphic study, this is defined by

Volume of parallelepiped = base are x height



1.3.3 The triple scalar product $A.(\boldsymbol{B} \times \boldsymbol{C})$

We now briefly discuss the scalar$\boldsymbol{A}.(\boldsymbol{B} \times \boldsymbol{C})$**.** This scalar represents the volume of the parallelepiped formed by the conterminous sides $A, B, C,$ since

$\boldsymbol{A}.(\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{ABC} sin\theta \cos a = hS$ =volume,

s being the area of the parallelogram with sides $\boldsymbol{B}$and C and h the height of parallelogram (Fig 1.9).

Now

$$A.(B \times C) = (A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3).\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_2 \end{vmatrix} \qquad (1.23)$$

$$= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1)$$

So that

$$A.(B \times C) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \qquad (1.24)$$

The exchange of two rows (or two columns) changes the sign of the determinant but does not change its absolute value. Using this property, we find

$$A.(B\times C)=\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}=-\begin{vmatrix}C_1 & C_2 & C_3\\ B_1 & B_2 & B_3\\ A_1 & A_2 & A_3\end{vmatrix}=C.(A\times B)$$

That is, the dot and the cross may be interchanged in the triple scalar product.

$$A.(B\times C)=(A\times B).C \qquad (1.25)$$

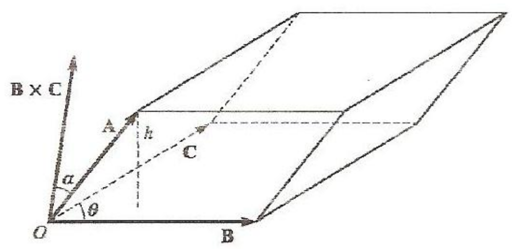


Figure 1.9. The triple scalar product of three vectors A, B, C

In fact, as long as the three vectors appear in cyclic order, $A\to B\to C\to A$, then the dot and cross may be inserted between any pairs:

$$A.(B\times C)=B.(A\times B). \quad (1.26)$$

It should be noted that the scalar resulting from the triple scalar product changes sign on an inversion of coordinates. For this reason, the triple scalar product is sometimes called a pseudo scalar.

1.3.4 The triple vector product

The triple product $A\times(B\times C)$ is a vector, since is the vector product of two vectors: A and $B\times C$.This vector perpendicular to $B\times C$ and so it lies in the plane $B$and C. If B is not parallel to $C. A\times(B\times C)=xB+yC$. Now dot both sides with A and we obtain $x(A.B)+y)A.C=0$, since $A.[A\times(B\times C)=0$.

Thus

$x/(A.C = -y/(A.B) \equiv \lambda(\lambda \text{is a scalar})$ (1.27a)

And so

$A \times (B \times C) = xB + yC = \lambda[B(A.C) - C(A.B)]$.(1.27b)

We now show that $\lambda = 1$.To do this, let us consider the special case when $B = A$ Dot the last equation with C:

$C \times [A \times (A \times C)] = \lambda[B(A.C) - C(A.B)]$.(1.28a)

Or, by an interchange of dot and cross

$-(A.C)^2 = \lambda[(A.C)^2 - A^2C^2]$.(1.28b)

In terms of the angles between the vector and their magnitudes the last equation becomes.

$$-A^2C^2sin^1\theta = \lambda(A^2C^2cos^2\theta - A^2C^2) = -\lambda A^2C^2sin^2\theta;$$

Hence$\lambda = 1.A$nd so

$$A \times (B \times C) = B(A.C) - C(A.B). \quad (1.29)$$

If a=10i – 3j + 5k, b= 2i + bj – 3k, and c= I + 10j -2k, verify that a.b + a.c = a. (b+c).

The position vector of a point is given by r=3t^2, + e$^{-t\,j}$ + 2 Cos 3tk, where t is time. Find the velocity and allele ratio vectors of the point.

Three masses, each of 1kg, have position vectors 2i + 3j, 6i + 4j. And 3i + 2j respectively. Find the position vector of the centre of gravity.

Forces f1 = 4i -6j + 5k, F2 = 03i + 2j + 3k, and F3 = 2i + 3j -6k all act on a particle which moved from r1 = 3i + 4j + 5k to r_2 = 2i – 3j + 4k, to r_3 = I + 4j – 2k. Find the work done on the particle in moving from r_1 to r_2, and from r_2 to r_3. What is the work done in moving from r_1 to r_3 ? Neglect gravity forces are given in Newton, distances in meters

Findaxb, exb and (cxa) xb if a=3i-2j + k, b=-I +j 2k and c=2i + 2j - 4k.

Verity that a.bxc and axb .c are equal using the vectors specifically in question (1).

(12)Show that the rectors a, b and c are coplanar, where a=3i + j - 2k, b=-5i +j-2k and c= - i - j – k

Find x if the vectors a,b and c are to be coplanar. A=3i -2j + 5k, b=2i –j + k and c=-x i + 2j – k.

A X B = AB Sin θe_n .C,Volume of parallelepiped

= A X B .C = C. A X B

= B X C. A = A.B X C

= C X A. B = B. C X A

We should note that in tripe product,

Ix i = j x j = kxk = 0

Ixj = 0 jxi = kxj,jxk =-k x j =I

And k x i =-ixk = j

Therefore to handle this easily, the students should be familiar with determine not. i.e

If A = qi + a2j + a3k; B= bii+b2j + b3k A XB =

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\left(a_2 b_3 - a_2 b_2\right) i - j\left(a_1 b_3 - a_3 b_1\right) + k\left(a_1 b_2 - a_2 b_2\right)$$

Example 6; find the area of the parallelogram with sides A = i+2j + 3k and B= 4i + 5j + 6k

$$A \times B = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = -3i + 6j - 3k$$

the area, A = $|A \times B| = (-3)^2 + (6)^2 + (-3)^2$

$= \sqrt{54}$

Example 7; find the value v of the parallelepiped with sides A = i+2j +3k,

B= 4i + 5j + 6k and c = 7i + 8j + 10k

A x B =-3i + bj – 3k as already determined therefore the volume of the parallelepiped is given by V = $|A.(B \; X \; C)| = |(A \; x \; B).C|$

$= |(-3i + 6j - 3k).(7i + 8j + 10k)|$

$= |= 21 + 48 - 30| = 3$

Triple product

This involves the products of three vectors

If A, B, C, are three vectors, the scalar formed by the product A. (B X C) is called the secular triple product. If A=

$a_1 i + a_2 j + a_3 k;$ B=$b_1 i + b_2 j + b_3 k$

$C = c_1 i + c_2 j + c_3 k,$ then

$$\text{Bxc}=\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\therefore \ \text{A.}(\text{B X C}) = \ a_1 i + a_2 j + a_3 k. \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$a_1 i + a_2 j + a_2 k;$ B=$b_1 i + b_2 j + b_3 k$

$C = c_1 i + c_2 j + c_3 k,$ then

$$\text{BxC}=\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\therefore \ \text{A.}(\text{B X C}) = \ a_1 i + a_2 j + a_3 k. \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

We not that in sealer triple products

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

A. (B x C)== $A.(CxB)$

$$B.(CxA) = \begin{vmatrix} a_1 & a_2 a_3 & \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = A.(BxC)$$

This means that interchange of two rows in a determinant reverses the sign of the product. This implies that the cyclic change of the vectors involved make the scalar triple product unchanged. Thus

A. (B x C)=B. (C X A)=C. (A X B)

Example 8: if a = I + 2j + 2k

B= 2i + 2j + k; C=3i + j + 2k

$$\begin{vmatrix} 1 & 2 & 2 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{vmatrix} =$$

$$1(6-1)-2(-4-3)+2(2+9)=41$$

$$C.(BXA)=\begin{vmatrix} 3 & 1 & -2 \\ 2 & -3 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

A. (B X C) $3(\text{-}6\text{-}2)-1(4-1)+(-2)(4+3)=-41$

but if we consider C.(A XB)

$$\begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 3(2+6) = 1(1-4)-2(-3-4)$$

$$=24+3+14=41$$

1.4.1 COPLANAR VECTORS

Scalar triple product provides a good test for coplaznarity of vectors: three vectors are coplanar if their triple product is zero ie if

A. (BX C) =0

We shall have to show by example as below.

Example 9

Show A = I + j -3k: B=2i-j + 2k and C=3i + j-k are coplanar

Solution

We need just to evaluate a (B x C)

A. (B X C) = $\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = 1(1-2)-2(-2-6)-3(2+3)$

$$=-1+16-15=0$$

Example 10

If 2i-j + 3k; B=3i + 2j k; C=I + pj + 4k are coplanar find the values of p

Solution

A. (BX C=0 = $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & p & 4 \end{vmatrix}$

$= 2(8-p) + 1(12-1) + 3(3p-2) = 0$

$\rightarrow p=3$

Reciprocal vectors

This is a concept that is particularly used in crystallography. Two sets of vectors A, B, C, and a, b, c, are called reciprocal set if

A. a = B. b = C. c =1

And a. B = a. c =b. a =b. C = c. B = c. B =0

The reciprocal vectors of a, b A C and are given by

$$A = \frac{b\ x\ c}{a.(b\ x\ c)}, B = \frac{c\ x\ a}{a.(b\ x\ c)}, C = \frac{a\ x\ b}{a.(b\ x\ c)}$$

Where a.(b x c) $\neq$ 0 meaning that reciprocal vectors exist if the vectors a, b and c are not coplanar. Again, if the vectors, then mutually orthogonal unit vectors, then it implies that a= a, B=b and C =0 so that the two system of vectors are identical

Example 11

Construct the reciprocal vectors of a = 2 i , b = j +k c= I +k

Solution

We evaluate first the triple scalar product a. (b x c)

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2(1-0) - 0(0-1) + 0(0-1) = 2$$

Therefore the reciprocal vectors

$$a = \frac{bxc}{2} = \frac{1}{2}\begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \frac{1}{2\left[i(1-0) - j(0-1) + k(0-1)\right]}$$

$$= \frac{1}{2}(i + j - k)$$

$$B = \frac{cxa}{2} = \frac{1}{2}\begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} = \frac{1}{2}\left[i(0) - j(0-2) + k(0)\right] = j$$

$$C = \frac{axb}{2} = \frac{1}{2}\begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \frac{1}{2\left[i(0) - j(2-0) + k(0)\right]} = -j$$

1.4.2 Distance from a point to a plane using scalar product

The direction of the shortest line from a point to a plane is parallel to the normal of the plane and this can be handled using scalar product as we intend to show in the example

Example 12

Find the distance of the point 3i + 2j + k from the plane (p-i-j.(i-j + k)=0

Find also the point on the plane nearest to the point 3i + 3j + k

Solution

Any point on the line through the point 3i + 3j + k with the direction of the vector I –j + k is

3i + 3j + k + (I – j + k=0 which intersects the plane when (3i + 3j + k + λ (i-j+k)-i-j). (i-j+k)=0

$\rightarrow$ (3+λ)i+(3-λ)j+(1+λ)k-i-j). (i-j+k)=0

3+λ-(3-λ)+ i+λ -1+1=0

2λ+1+λ=0

$3\lambda + 1 = 0 \rightarrow \lambda = \frac{1}{3}$

The distance of the point from the plane is

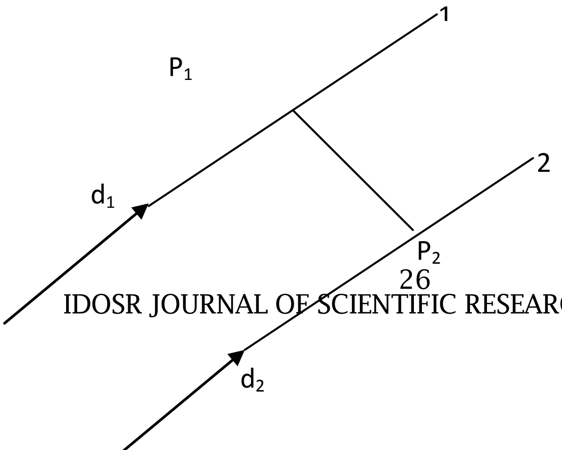
$$\frac{1i - j + k}{3} = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{3}{9}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

The point on the plane is 3i +3j + k + λ (I –j + k)=3i + 3j + k-$\frac{1}{3}$(I - j + k)

$$= \frac{8}{3}i - \frac{10}{3}j + \frac{2}{3}k$$

1.4.3 Shortest distance between two lines by scalar product

The line of shortest distance between two lines is perpendicular to both of the lines



Here we not that p_2 p_2 is perpendicular to both d_1 and d_2

Thus

p_2p_1. d_1 = p_2 p_1. d_2 =0 if p_1 and p_2 are to be the nearest points to lines (1) and (2)

Example 13

Two lines are given by l_1 = 2i + 2j -5k + λ (3i + 2j -4k)

L_2 = 3 i - j + 4k + λ' (5 I -2j – 4 k)

Where λ and λ' are variable find

1. The shortest distance between
2. The values of λ and λ' corresponding to the points of closest approach
3. The position vectors of the points of closest approach

Solution

d_1 =3i +2j -4k and d_2=5i-2j-4k

Where d_1 and d_2 are vector parallel to the directions of l_1 and l_2. Let d=ai + bj + ck be a vector with direction of the line.

$\therefore$ (d- d_1). (d-d_2) =0 (ai + bj + ck). (3i + 2j -k)= (ai + bj + ck). (5i – 2j -4k)

3a + 2b –c =05c – 2b-4c =0

Solving the two equations will yield a = c and b=$\frac{a}{2}$

Therefore

$$d = \frac{2ai + aj + 2ak}{2} = [2i + j + 2k]$$

$$|d| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

$$e_d = \frac{d}{|d|} = \frac{2i + j + 2k}{2}$$

Where d is the direction of the line of shortest distance between l_1 and l_2 the position of a point on l_1 relative to a point on l_2 is

(l_2 - l_1) $\lambda - \lambda^1 = 0$ recalling that d_1d=0

=3i + j + k4 + 0 (5i - 2j – 4k) =

$[2i + 3j - 5k] + 0(3i + 2j - 4k) = i - 4j + 9k$

Ed. ($L_2 - L_1$) $\lambda - \lambda^1 = 0$ is the shortest distance between L_1 and l_2

$= (i - 4j + 9k).\ \frac{2i + j + 2k}{3} = \frac{2 - 4 + 9}{9} = \frac{16}{3}$

Vector triple product of three vectors If A, B and C are three vectors, then Ax (B X C) and (A X B) XC are called the vector triple products the result means that B X c is a vector perpendicular to the plane of B and C and the vector A X (B CX C) is perpendicular to the plane containing a and B and C If we consider a = $a_1i + a_2j + a_3k$

B= $b_1i + b_2j + b_3k$ and C=$c_1i + c_2j + c_3k$

Then B X C = $\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = i \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - j \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + k \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

Therefore A X (B X C)

$= \begin{vmatrix} i \\ a_1 \\ b_2 \quad b_3 \\ c_2 \quad c_3 \end{vmatrix} \quad \begin{vmatrix} j \\ a_2 \\ b_1 \quad b_3 \\ c_1 \quad c_3 \end{vmatrix} \quad \begin{vmatrix} k \\ a_3 \\ b_1 \quad b_2 \\ c_1 \quad c_3 \end{vmatrix}$

$= i\ a_1(b_2\ c_3 = b_3\ c_2) = j\ \ a_2(b_1\ c_3 - c_3 b_1) + k\ a_3(b_1\ c_2 - b_2\ c_1)$

Example 15

Determine the vector triple product A x (B X C) if a = 2i – 3j + ki

B= 2i + 3j – 2k, c= 3i + j +4k

Solution

B X C = $\begin{vmatrix} i & j & k \\ 2 & 3 & -2 \\ 3 & 1 & 4 \end{vmatrix} = i\ (12+2) - j\ (8+6) + k\ (2-6)$

= 14j - 14j - 4k

for a X C (B X C) = $\begin{vmatrix} 1 & j & k \\ 2 & -3 & 1 \\ 14 & -14 & -4 \end{vmatrix}$ =

$$i(12+14)-j(-8-14)+k(-28+42)$$

$$Ax(BXC)=28i+22j+14k=2\ (14i+11j+7k)$$

There is an easier way of determining a vector triple product which can be proved to have relationship such as

A X(B X C)= (A .C)B- (A. B)C and (A X b) X C= (C. A)B-(C.B)A

We can show this and we wish to consider that

A X (B X C)= (A.C) B- (A. B)C

Let A = $a_1 i + a_{2\,j} + a_3 k \Rightarrow$

$$(a_1, a_2, a_3)$$

$$B = b_1 i + b_2 j + b_3 k \to (b_1, b_2, b_3)$$

$$C = c_1 i + c_2 j + c_3 k \to (c_1, c_2, c_3)$$

From the previous example

$$\begin{aligned} &= \left[b_1 (a_1c_1 + a_2c_2 + a_3c_3) c_1 (a_1b_1 + a_2b_2 + a_3b_3) \right] i \\ &+ \left[b_2 (a_1c_1 + a_2c_2 + a_3c_3) - c_2 (a_1b_1 + a_2b_2 + a_3b_3) \right] j \\ \text{B X C} = &+ \left[b_3 (a_1c_1 + a_2c_2 + a_2b_2) - c_3 (a_1b_1 + a_2b_2 + a_3b_3) \right] k \\ &= (a_1c_1 + a_2c_2 + a_2c_3)(b_1 i + b_2 j + b_3 k) \\ &- (a_1b_1 + a_2b_2 + a_3b_3)(c_1 i + c_2 j + c_3 k) \end{aligned}$$

Say equal D then A X (B X C)= A X D

$$= (a_2d_3 - a_3d_2, a_2d_1 - a_1d_3, a_1d_2 - a_2d_1)$$

$$= (a_2d_3 - a_3d_2) i + (a_3d_1 - a_1d_3)\ j + (a_1d_2 - a_2d_1) k$$

$$= \Sigma (a_2d_3 - a_3d_2) i$$

$$= \sum (a_2 (b_1c_2 - b_2c_1) - a_3 (b_3c_1 - b_1c_3))\ i$$

$$\sum [a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 + a_1b_1c_1 + a_1b_1c_1]\ \mathrm{i}$$

By adding and subtracting $a_1b_1c_1$

$$=\left[b_1\left(a_1c_1+a_2c_2+a_3c_3\right)c_1\left(a_1b_1+a_2b_2+a_3b_3\right)\right]i$$
$$+\left[b_2\left(a_1c_1+a_2c_2+a_3c_3\right)-c_2\left(a_1b_1+a_2b_2+a_3b_3\right)\right]j$$
$$+\left[b_3\left(a_1c_1+a_2c_2+a_3b_3\right)-c_3\left(a_1b_1+a_2b_2+a_3b_3\right)\right]k$$
$$=\left(a_1c_1+a_2c_2+a_3c_3\right)\left(b_1i+b_2j+b_3k\right)$$
$$-\left(a_1b_1+a_2b_2+a_3b_3\right)\left(c_1i+c_2j+c_3k\right)$$

<u>(A. C) B - (A . B) C.</u>

Example 16

If a = I + 3j + 2k; B= 2i + 5j –k

C = I + 2j + 3k

Show that A X (B X C)= (A.C)B- (A.B)C

(A X B) X C = (C . A) B – (C.B) A

A. C = 1 + 6 + 6=13

(A.C) B = 26i + 65j-13k

A.B = 2 + 15-2=15

(A.B) C =15i + 30j + 45k

:.(A.C) B -(A.B) C = (26i + 65j - 13k) - (15i + 30j + 45k)

=11i + 35j + 58k

Now

(C.A) B= (1+ 6 + 6) (2i + 5j -k) = 13 (2i+ 5j – k

(C.B) A= (2 + 10 -3) (I + 3j 2k)= 9 (I + 3j + 2k)

:. (C +B) B - (C .B) A = 26i + 65j -13k +-(9i + 27j + 18k)

=17i + 39j + 31k

These results shows that

AXC (B X C) $\neq$ (A X B) XC.

1.5.1 Change of coordinate system

Vector equations are independent of the coordinate system we happen to use. But the components of a vector quantity are different in different coordinate systems. We now make a brief study of how to represent a vector in different coordinate systems. As the rectangular Cartesian coordinate system is the basic type of coordinate system, we shall limit our discussion to it. Consider the vector A expressed in terms of the unit coordinate vector $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$.

$$A = A'_1 \hat{e}'_1 + A'_2 \hat{e}'_2 + A' \hat{e}'_3 = \sum_{i=1}^{3} A_1 \hat{e}_1 \qquad (1.30)$$

Relative to a new system $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ that has a different orientation from that of the old system $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, vector A is expressed as

$$A = A'_1 \hat{e}'_1 + A'_2 \hat{e}'_2 + A' \hat{e}'_3 = \sum_{i=1}^{3} A_1 \hat{e}_1$$

Note that the dot product $A.\hat{e}'_1$ is equal to A'_1, the projection of A on the direction of $.\hat{e}'_1 A.\hat{e}'_2$ is equal to $.\hat{e}'_2$ and $A..\hat{e}'_3$ is equal to $.\hat{e}'_3$. Thus we may write.

$$\left.\begin{aligned} A'_1 &= (\hat{e}'_1.\hat{e}'_1)A_1 + (\hat{e}'_2.\hat{e}'_1)A_2 + (\hat{e}'_3.\hat{e}'_1)A_3 \\ A'_2 &= (\hat{e}'_1.\hat{e}'_2)A_1 + (\hat{e}'_2.\hat{e}'_2)A_2 + (\hat{e}'_3.\hat{e}'_2)A_3 \\ A'_3 &= (\hat{e}'_1.\hat{e}'_3)A_1 + (\hat{e}'_2.\hat{e}'_3)A_2 + (\hat{e}'_3.\hat{e}'_3)A_3 \end{aligned}\right\} \qquad (1.31)$$

The dot product $(\hat{e}'_1.\hat{e}'_1)$ are the direction cosines of the axes of the new coordinate system relative to the old system $\hat{e}'_1.\hat{e}'_1 = \cos(x_i^i, x_j)$; they are often called the coefficients of transformation. In matrix notation, we can write the above system of equations as

The 3×3 matrix in the above equation is called the rotation (or transformation) matrix, and is an orthogonal matrix. One advantage of using a matrix is that successive transformations can be handled easily by means of matrix in multiplication.

A matrix is an ordered array of scalars that obeys prescribed rules of addition and multiplication. A particular matrix element is specified by its row number followed by its column number. Thus a_{ij} is the matrix element in the ith row and jth column. Alternative ways of representing matrix $\bar{A}$ are $[a_{ij}]$ or the entire array.

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \cdots & \cdots & \cdots \; \cdots \\ a_{m1} & a_{m2} & \cdots a_{mn} \end{pmatrix} \qquad (1.32)$$

is matrix. A vector is represented in matrix for by writing its components as either a row or column array.

The multiplication of a matrix and a matrix is defined only when the number of columns of column is equal to the number of rows and is performed in the same way as the multiplication of two determinants.

1.5.2 The linear vector space

We have found that it is very convenient to use vector components, in particular, the unit coordinate vector $\hat{e}_i$ (i= 1,2,3). The three unite vector $\hat{e}_i$ are orthogonal and normal, or, as we shall say, orthonormal. This orthonormal property is conveniently written as$Eq.$(1.8). But there is nothing special about these orthonormal unit vectors $\hat{e}_i'$. If we refer the components of the vectors to a different system of rectangular coordinates, we need to introduce another setoff three orthonormal unit vector $\hat{f}_1, \hat{f}_2,$ and $\hat{f}_3$:

$$\hat{f}_i \hat{f}_j = \delta_{ij} \, (i, j = 1,2,3). \qquad (1.33a)$$

For any vector A we now write

$\sum_{i=1}^{3} c_i \hat{f}_i$, and $c_i = \hat{f}_i \, . \boldsymbol{A}$.n (1.33b)

We see that we can define a large number of different coordinate systems. But the physically significant quantities are the vectors themselves and certain functions of these, which are independent of the coordinate system used. The orthonormal condition is convenient is practice. If we also admit oblique Cartesian coordinates then the $\hat{f}_i$ need neither be normal nor orthogonal; they could be any three non-coplanar vectors, and any vector Acan still be written as a linear superposition of the $\hat{f}_i$.

$$A = c_i \hat{f}_i, + c_2 \hat{f}_2 + c_3 \hat{f}_3 \qquad (1.34)$$

Starting with the vector $\hat{f}_i$, we can find linear combinations of them by the algebraic operations of vector addition and multiplication of vectors by scalars, and then the collection of all such vector makes up the three-dimensional linear space often called V_3($V$of vector)or R_3($R$for real) or E_3($E$for Euclidean). The vector $\hat{f}_1, \hat{f}_2, \hat{f}_3$ are called the base vectors or bases of the vector spaceV_3. Any set of vectors, such as the $\hat{f}_1$, which can serve as the bases or base vector of V_3 is called complete, and we say it spans the linear vector space. The base vector are also linearly independent because no relation of form

$$c_i \hat{f}_i, + c_2 \hat{f}_2 + c_3 \hat{f}_3 = 0 \qquad (1.35)$$

Exists between them, unless $c_1 = c_2 = c_3 = 0$.

The notion of a vector space is much more general than the real vector space V_3. Extending the concept of V_3, it is convenient to call an ordered set of n matrices, or functions, or 'vector' (or an n −vector) in n −dimensioal space V_n Chapter 5 well provide justification for doing this. Taking cue from V_3, vector addition in V_n is defined to be to be

$$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n) \qquad (1.36)$$

And multiplication by scalars is defined by

$$a(x_1, \cdots, x_n) = (ax_1, \cdots, ax_n) \qquad (1.37)$$

Where a is real. With these two algebraic operations of vector addition and multiplication by scalars, we call V_n a vector space. In addition to this algebraic structure, V_n has geometric structure derived from the length defined to be

$$\left(\sum_{j=1}^{n} x_j^2\right)^{1/2} = \sqrt{x_1^2 + \cdots + x_n^2} \qquad (1.38)$$

The dot product of two n −vectors can be defined by

$$(x_1, \cdots, x_n).(y_1, \cdots, y_n) = \sum_{j-1}^{n} x_j y_j \qquad (1.39)$$

In V_n, vectors are not directed line segments as in V_3; they may be an ordered set of n operators, matrices, or functions. We do not want to become sidetracked from our main goal of this chapter, so we end out discussion of vector space here.

1.5.3 Vector differentiation

Up to this point we have been concerned mainly with vector algebra. A vector may be a function of one or more scalars and vectors. We have encountered, for example, many important vectors in mechanics that are functions of time and position variables. We now turn to the study of the calculus of vectors.

Physicists like the concept of field and use it to represent a physical quantity that is a function of position in a given region. Temperature is a scalar field, because its value depends upon location: to each point (x, y, z) is associated a temperature $T(x, y, z)$. The function $T(x, y, z)$ is a scalar field, whose value is a real number depending only on the point in space but not on the particular choice of the coordinate system. A vector field, on the other hand, associates with each point a vector (that is, we associate three numbers at each point), such as the wind velocity or the strength of the electric or magnetic field. When described in a rotated system, for example, the three components of the vector associate with one and the same point will change in numerical value. Physically and geometrically important concepts in connection with scalar and vector fields are the gradient, divergence, curl, and the corresponding integral theorems.

The basic concepts of calculus, such as continuity and differentiability, can be naturally extended to vector calculus. Consider a vector, $\boldsymbol{A}$ whose components are function of a single variable u. If the vector $\boldsymbol{A}$represents position or velocity, for example, then the parameter u is usually time t, but is can be any quantity that determines the components of $\boldsymbol{A}$. If we introduce a Cartesian coordinate system, the vector function $\boldsymbol{A}(u)$ may be written as

$$A(u) = A_1(u)\hat{e}_1 + A_2(u)\hat{e}_2 + A_3(u)\hat{e}_3 \tag{1.40}$$

$A(u)$is said to be continuous at $u = u_0$ if it is defined in some neighborhood of u_0 and

$$\lim_{n\to u_0} A(u) = A(u_0) \tag{1.41}$$

Note that $A(u)$

Is continuous at u_0 if and only if its three components are continuous at u_0.

$A(u)$is said to be differentiable at a point u if the limit

$$\frac{dA(u)}{du} - \lim_{\Delta u\to 0}\frac{A(u+\Delta u) - A(u)}{\Delta u} \tag{1.42}$$

Exists. The vector $A^1(u) = dA(u)/du$ is called the derivative of $A(u)$; and to differentiate a vector function we differentiation each component separately:

$$A'(u) = A'_1(u)\hat{e}_1 + A'_2(u)\hat{e}_2 + A'_3(u)\hat{e}_3 \tag{1.43a}$$

Note that the unit coordinate vectors are fixed in space. Higher derivatives of $A(u)$ can be similarly defined.

If $\boldsymbol{A}$is a vector depending on more than one scalar variable, say u, v for example, we write$\boldsymbol{A} = \boldsymbol{A}(u, v)$. Then

$$dA = (\partial A/\partial u)du + (\partial A/\partial v)dv \tag{1.44}$$

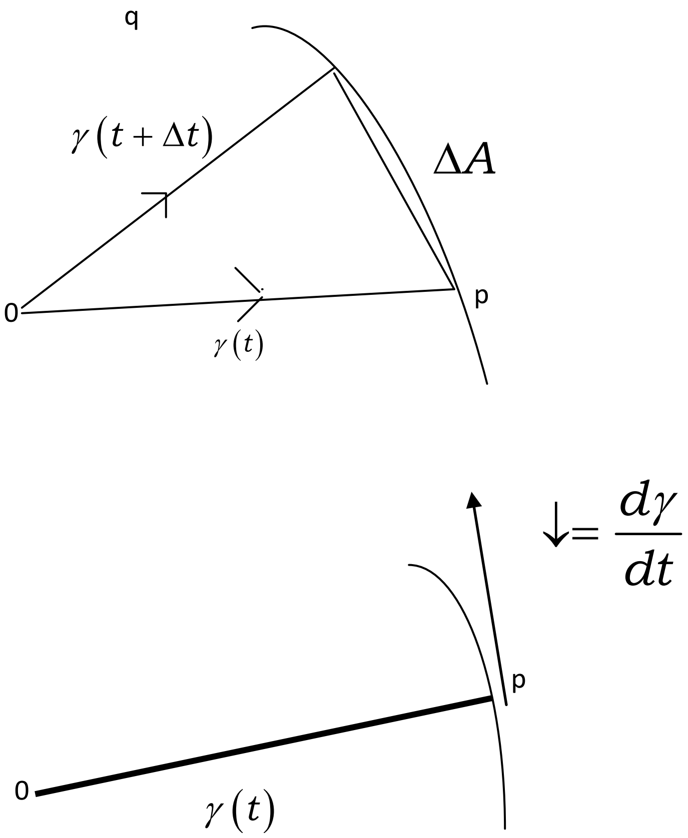
Is the differential of A and

$$\frac{\partial A}{\partial u} - \lim_{\Delta u\to 0}\frac{A(u+\Delta u, v) - A(u, v)}{\partial u} \tag{1.45}$$

And similarly for $\frac{\partial A}{\partial u}$

Derivatives of products obey rules similar to those for scalar functions. However, when cross products are involved the order may be important. This can be elucidated by the concept of unit tangent vectors

1.5.4 Unit tangent vectors



Fig,1.10 unit tangent

Form this diagram it is seen that if QP moves to $\perp$ at P where t

becomes t + Δt, then as $\Delta t, \to 0$,

The direction of the chord $\overline{PQ}$ becomes the tangent to the curve at p.

Q this means that the direction of $\frac{dr}{dt}$ is along the tangent to the locus

P therefore the direction of the vector denoting $\frac{d}{dt}r(t)$ is parallel to

the tangent to the curve at P_1

Then the unit tangent vector $\perp$ at P can be obtain from

$$\frac{\frac{d[r(t)]}{dt}}{\left|\frac{d[r(t)]}{dt}\right|} = \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|}$$

Example 4

Obtain the unit tangent vector at the point (0, 3,5) for the curve with parametric equation x=2t; $y = t^2 + 3, z = 2t^2 + 5$

Solution

We see that the point (0,3,5) corresponds to t=o,

$\therefore r = a_x i$ + a_y j + a_z k= 2ti + (t^2+3)j + $\left(2t^2 + 5\right)k$

$$\frac{dr}{dt} = 2i + 2tj + 4tk$$

$$\left.\frac{dr}{dt}\right|_{t=0} = 2i$$

$$hence \left|\frac{d\gamma}{dt}\right| = \sqrt{2^2} = 2$$

$$\therefore \perp = \frac{d\gamma}{dt} / \left|\frac{d\gamma}{dt}\right| = \frac{2i}{2} = i$$

Example 5

Determine the unit tangent vector for the curve atthe point [6, 8, 10]

$$x = 3t, y = t^3, z = 2t^2 + 2t$$

$$\gamma = 3ti + t^2 j + \left(2t^2 + 2t\right)k$$

$$\gamma' = 3i + 3t^2 j + \left(4t + 2\right)k$$

$$\gamma' /_{t=2}; = 3i + 12j + 10k$$

$$|\gamma'| = (9 + 144 + 100)^{1/2} = \sqrt{253}$$

$$\therefore \perp = \frac{\gamma'}{|\gamma'|} = \frac{1}{\sqrt{253}}\left(3i + 12j + 10k\right)$$

If a vector R is a function of two independent variable u and d v, then the rules of differentiation in relation to the partial differentiation follow the normal pattern.

If for instance $\phi\ R = xi + yj + \neq k$, then x, y, $\neq$ will also be functions of U and V than

$$\frac{\partial R}{\partial u} = \frac{\partial x}{\partial \mathrm{u}} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} K$$

$$\frac{\partial^2}{\partial u^2} = i \frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 y}{\partial u^2} j \frac{\partial^2 z}{\partial u^2} k$$

$$\frac{\partial R}{\partial V} = \frac{\partial x}{\partial V} i + \frac{\partial y}{\partial V} j + \frac{\partial z}{\partial V} k$$

$$\frac{d^2 R}{\partial V^2} = \frac{\partial^2 x}{\partial V^2} i + \frac{\partial^2 y}{\partial V^2} j + \frac{\partial^2 z}{\partial V^2} k$$

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{\partial^2 x}{\partial u \partial v} i + \frac{\partial^2 y}{\partial u \partial v} j + \frac{\partial^2 z}{\partial u \partial v} K$$

Example 6

If $R = 2u^2 vi + \left(\mathrm{u}^2 + 2v^2\right) j + \left(\mathrm{u} + zv^2\right) k$, obtain the second order derivative.

Solution

$$\frac{\partial R}{\partial u} = 4uvi + \left(2U + 2V^2\right) j + \left(1 + 2V^2\right) k$$

$$\frac{\partial R}{\partial V} = Zu^2 i + R = 2u^2 vi + \left(\mathrm{u}^2 + 4V\right) j + \left(\mathrm{u} + 4V\right) k$$

$$\frac{\partial^2 R}{\partial u \partial V} = 4ui + R = 2u^2 vi + \left(2u + 2V\right) j + \left(i + 4V\right) k.$$

This is so straight forward. One can even continue on to higher order partial derivative

1.6.1 Vector integration

The process involved in integration is the reverse of that for differentiation. The integration procedure is the same with that of scalar integration. If for instance a vector $R = xi + yj + zk$ where R, x, y, z are expressed as function of U, then

$$\int_a^b R du = i \int_a^b x du + jk \int_a^b du \qquad (1.46)$$

Example 7

Find the integral of the vectors function find as

$R = (2t^2 + 4t)i + (3t^2 - 5)j + 4t^2 k$ for 1 to 3

Solution

$$\int_1^3 R dt = i \int_1^3 (2t^2 + 4t)\, dt + j \int_1^3 (3t^2 - 5) dt + k \int_1^3 4t^3 dt$$

$$= \left[i\left(\frac{2}{3}t^3 + 2t^2\right) + j(t^3 - 5t) + k(t^4) \right]_1^3$$

$$\frac{100}{3} i + 16j + 80k$$

Example 8

If
$$R = 4ui + 2u^2 j + (u^2 + 2)k \ and$$
$$B = 2ui - 4uj + (u - 3)k$$

Evaluate $\int_0^1 (RxB)\,du$

Solution

We first determine RXB in terms of U

$$RXB = \begin{vmatrix} i & j & k \\ 4u & 2u^2 & u^2+2 \\ 2u & -4u & u-3 \end{vmatrix}$$

$$= i(2u^3 - 6u^2 + 4u^3 8u) - j(4u^2 - 12u - 2u^3 + 4u) + K(-16u^2 + 4u^3)$$

$$= (-6u^3 - 6^2 + 8u)i - j(-2u^3 + 4u^2 - 8u) - K(4u^3 + 16u^2)$$

$$\therefore \int_0^1 (RXB)\,du = i\int_0^1 (6u^3 - 6u^2 + 8u)\,du - j\int_0^1 (-2u^3 + 4u^2 - 8u)\,du - k\int_0^1 (4u^3 + 16u^2)\,du$$

$$= [i(3/2u^4 - 3u^3 + 4u^2)]_0^1 \int_0^1 -j\left(-\frac{1}{2}4u^4 + \frac{4}{3}u^3 - 4u^2\right)$$

$$= i(3 - 2 + 4) - j\left(-\frac{1}{2} + \frac{4}{3} - 4\right) - K\left(1 + \frac{16}{3}\right)$$

$$= 5\ i - \frac{11}{12}j - \frac{19}{3}K.$$

Note that if the integration involves triple vector product say AX (BXC), it will be easier for the reader to use the relation

AX (BXC)=(A.C) B- (A.B) C

And now integrate as show below.

$$\int_a^b AX\,(BXC)\,dt = \int_a^b (A\,.\,C)\ \text{B dt} - \int_a^b (A\,.\,B)\ \text{C dt.}$$

$$or = \int_a^b \left[(A\,.\,C) - (A\,.\,B)\ C\right]\ dt.$$

1.6. 2 Grad div and curl

In physics, field is explained as a region within which some physical agency act or experienced such as magnetic force, electric force, temperature, potential density.

Among these, some are vectors while some are scalars.

The point say $(x,\ \text{y,z})$ in a region of a vector field may be associated to the vector function F $(x,\ \text{y, z})$ similarly point P $(x,\ \text{y,z})$ in a region of scalar field may be associated to scalar function $\phi\ (x,\ \text{y. z})$

If a scalar function $\phi\ (x,\ \text{y. z})$ is differentiable with respect to its variables $(x,\ \text{y.}\ \neq)$ throughout the region, then the gradient of ϕ is simply written as grad ϕ and defined as

$$\text{grad}\ \phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}\text{j} + \frac{\partial\phi}{\partial z}\text{K} \qquad (1.47)$$

While ϕ is a scalar function, grad ϕ is a vector function.

$\frac{\partial}{\partial x}(2x^2y^2z^3)+j\frac{\partial}{\partial y}(2x^2y^2z^3)$ Is a differential operator, ∇ call 'del' i.e ∇

$\equiv i\ \frac{\partial\phi}{dx}+j\frac{\partial\phi}{dy}+K\frac{\partial\phi}{\partial}$

Example 8

If ϕ = $2x^2y^2z^3$ and depends upon the position p

Grad ϕ = $i\frac{\partial}{\partial x}(2x^2y^2z^3)+j\frac{\partial}{\partial y}(2x^2y^3z^3)+K\,\partial/\partial z\,[2x^2y^2z^3$

$\nabla\phi=$ grad ϕ=4xy^2z^3i + 4x$^2yz^3j$ + 6x$^2y^2z^2k$.

Divergence of a vector A is a written as div A,

Therefore, if $a_xi+a_yja_zk$

and $\nabla=i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}$ then

div $A=\nabla.A=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left(a_xi+a_yj+a_zk\right)$

$=\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}.$

Example 12

If $A=x^2yi+x^yz^2j+y^2z^2k$

Div $=i\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)+j\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)+k\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)$

$=2xy+xz^2+2y^2z$

Curl of a vector function, (curl)

Curl means ∇x which gives and vector result. For instance if $A=a_xi+a_yj+a_zk$

Then curl $\text{A}=\nabla\times\text{A}$

$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(a_xi+a_yj+a_zk\right)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$\therefore \nabla \times A = i\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) j\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + k\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$= i\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + j\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + k\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

Then procedure here involves determinant method that common in matrices problem.

Example 9

Find CurlAat the point (1,0-3) given that

A = $ze^{x^2} yi + 2xz^2 \cos yj + (x^2 + 2y)k$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{x^2 y} & 2xz^2 \cos y & x^2 + 2y \end{vmatrix}$$

=

$$i[\frac{\partial}{\partial y}(x^2 y) - \frac{\partial}{\partial z}(2xz^2 \cos y)] - j[\frac{\partial}{\partial x}(x^2 + 2y) - \frac{\partial}{\partial z}(ze^{x^2 y})] + k[\frac{\partial}{\partial x}(2xz^2 \cos y) - \frac{\partial}{\partial}(zx - e^{x^2 y})]$$

= $i(2 - 4xzcosy) - j2x - e^{x^2 y}) + k(2z \cos y - ze^{x^2 y})$

Is Curl A

At the point (1, 0,-3)

Curl A = $\nabla xA = 14i - j + 21k$

Here we summarize in this case a situation where are multiple functions such Curl A gradϕ where ϕ is a scalar function;

gradϕ= $\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$

$$curl\,grad\phi = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = i[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}] - j[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}] + k[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}]$$

Curl gradϕ= $\nabla x[\nabla\phi] = 0$

1.6.3 Space curves

As an application of vector differentiation, let use consider some basic facts about curves in space. If $A(u)$ is the position vector $r(u)$ joining the origin of a coordinate system and any point $P(x_1, x_2, x_3)$ in space as shown in Fig 1.10, then we have

$$r(u) = x_1'(u)\hat{e}_1 + x_2'(u)\hat{e}_2 + x_3'(u)\hat{e}_3 \quad (1.48)$$

As u changes, the terminal point P or r describes a curve C in space. Eq. (1.35) is called a parametric representation of then curve C, and u is the parameter of this representation.
Then

$$\frac{\Delta r}{\Delta u}\left(= \frac{r(u+\Delta u) - r(r)}{\Delta u}\right)$$

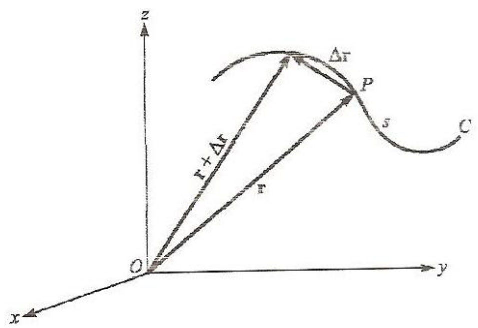


Figure 1.11 parametric representation of a curve

Is a vector in the direction of Δr, and its limit (if it exists) dr/du is a vector in the direction of the tangent to the curve at (x_1, x_2, x_3). If u is the are length s measured from some fixed point on the curve C, then $\frac{dr}{ds} = \hat{T}$ is a unit tangent vector to the curve$C$. The rate at which $\hat{T}$ changes with respect to s is a measure of the curvature of C and is given by $d\hat{T}/ds$ at any given point on C is normal to the curve at that point $\hat{T}.\hat{T} = 1, d(\hat{T}.\hat{T})/ds = 0$, from this we get $\hat{T}.\hat{T}/ds = 0$, so they are normal to each other. If $\hat{N}$ is a unity vector in this normal direction (called the principle normal to the curve), then $\frac{d\hat{T}}{ds} = k\hat{N}$, called then radius of curvature. In physics, we often study the motion of particles along curves, so the above results may be of value.

In mechanics, the parameter u is time t, then $\frac{dr}{dt} = v$ is the velocity of the particle which is tangent to the curve at the specific point. Now we can write

$$v = \frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = v\hat{T} \qquad (1.49)$$

Where $v$the magnitude is called the speed. Similarly, $a = dv/dt$ is the acceleration of the particle.

1.6.4 Motion in a plane

Consider a particle P moving in aplane along a curve C (Fig.1.12). now$r\hat{e}_r$, where $\hat{e}_r$ is a unit vector in the direction of r. Hence

$$v = \frac{dr}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

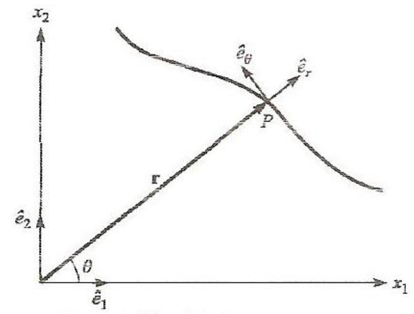


Figure 1.12 Motion in a plane

Now $d\hat{e}_r/dt$ is perpendicular to $\hat{e}_r$. Also $|\ d\hat{e}_r|dt| = d\theta/dt$; we can easily verify this by differentiating $\hat{e}_r = cos\theta\hat{e}_1 + sin\theta\hat{e}_2$. Hence.

$$v = \frac{dr}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_{\theta;} \qquad (1.50\_$$

$\hat{e}_{\theta;}$is a unit vector perpendicular to $\hat{e}_{r;}$

Differentiating again we obtain

$$a = \frac{dv}{dt} = \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + r\frac{d^2\theta}{dt^2}\hat{e}_\theta + r\frac{d\theta}{dt}\hat{e}_\theta$$

$$= \frac{d^2r}{dt^2}\hat{e}_r + 2\frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + r\frac{d^2\theta}{dt^2}\hat{e}_r - r\left(\frac{d\theta}{dt}\right)^2\hat{e}_r\left(\because \frac{d\hat{e}_r}{dt} = -\frac{d\theta}{dt}\hat{e}_r\right)$$

Thus

$$a = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\hat{e}_\theta. \qquad (1.51)$$

1.6.5 A vector treatment of classical orbit theory

To illustrate the power and use of vector methods, we now employ them to work out the Keplerian orbits. We first prove Kepler's second law which can be stated as: angular momentum is constant in a central force field. A central force is a force whose line of action passes through a single point or center and whose magnitude depends only on the distance from the center. Gravity and electrostatic forces are central forces. A general discussion on central force can be found in, for example, Chapter 6 of classical mechanics, Tai L. Chow, John Wiley, New York, 1995.

Differentiating the angular momentum $L = r \times P$ with respect to time, we obtain.

$$\frac{dL}{dt} = \frac{dr}{dt} \times P + r \times \frac{dp}{dt}. \quad (1.52)$$

The first vector product vanishes because $P = mdr/dt$ so dr/dr and P are parallel. The second vector product is simply $r \times F$ by newton's second law, and hence vanishes for all forces directed along the position vector, r that is, for all central forces. Thus the angular momentum L is a constant vector in central force motion. This implies that the position vector r, and therefore the entire orbit, lies in fixed plane in three-dimensional space. This result is essentially Kepler's second law, which is often stated in terms of the conservation of area velocity, $|L|/2m$.

We now consider the inverse-square central force of gravitational and electrostatics. Newton's second law then gives

$$\frac{mdv}{dt} = -\left(\frac{k}{r^2}\right)\hat{n} \quad (1.53)$$

Where $\hat{n} = r/r$ is a unit vector in the r −directon, and $k = Gm_1m_2$ for the gravitational force, and $k = q_1q_2$ for the electrostatic force in egs units. First we note that

$$v = \frac{dr}{dt} = \frac{dr}{dt\hat{n}} + \frac{rd\hat{n}}{dt}. \quad (1.54)$$

Then L becomes

$$L = r \times (mv) = mr\ ^2[\hat{n} \times (d\hat{n}/dt)] \quad (1.55)$$

Now consider

$$\frac{d}{dt}(v \times L) = \frac{dv}{dt} \times L = -\frac{k}{mr^2}(\hat{n} \times L) = -\frac{k}{mr^2}\left[\hat{n} \times mr^2\left(\hat{n} \times \frac{d\hat{n}}{dt}\right)\right]$$

$$= -k[\hat{n}\left(\frac{d\hat{n}}{dt}.\hat{n}\right) - \left(\frac{d\hat{n}}{dt}\right)(\hat{n}.\hat{n})]$$

Since $\hat{n}.\hat{n} = 1$, it follows by differentiation the $\hat{n}.\frac{d\hat{n}}{dt} = 0$. Thus we obtain

$$\frac{d}{dt}(v \times L) = \frac{kd\hat{n}}{dt};$$

Integration gives

$$v \times L = k\hat{n} + C \qquad (1.56)$$

Where C is a constant vector? It lies along, and fixes the position of, the major axis of the orbit as we shall see after we complete the derivation of the orbit. To find the orbit, we form the scalar quantity

$$L^2 = L(r \times mv) = mr.(v \times L) = mr(k + C\, cos\theta), \qquad (1.57)$$

Where θ is the angle measured fron C (which we may take to be the x −axis)to$r$. Solving for r, we obtain

$$r = \frac{L^2/km}{1 + C/(k\, cos\theta)} = \frac{A}{1 + \varepsilon\, cos\theta} \qquad (1.58)$$

Eq. (1.40) is a conic section with one focus at the origin, where ε represents the eccentricity of the conic section; depending on its values, the conic section may be a circle, a ellipse, as parabola, or a hyperbola. The eccentricity can be easily determined in terms of the constants of motion:

$$\varepsilon = \frac{C}{k} = \frac{1}{k}|(v \times L) - k\hat{n}|$$

$$= \frac{1}{k}[|v \times L|^2 + k^2 - 2k\hat{n}.(v \times L)]^{1/2}$$

Now $|v \times L|^2 = v^2L^2$ because v is perpendicular to L. Using Eq (1.58), we obtain

$$\varepsilon = \frac{1}{k}\left[v^2L^2 + k^2 - \frac{2kL^2}{mr}\right]^{1/2} = \left[1 + \frac{2L^2}{mk^2}\left(\frac{1}{2}mv^2 - \frac{k}{r}\right)\right]^{1/2} = \left[1 + \frac{2L^2E}{mk^2}\right]^{1/2}$$

$E$is the constant energy of the system.

1.6.6 Vector differentiation of a scalar field and the gradient

Given a scalar field in certain region of space given by a scalar function $\emptyset(x_1, x_2, x_3)$ that is defined and differentiable at each point with respect to the position coordinates(x_1, x_2, x_3), the total differential corresponding to an infinitesimal change $dr = (dx_1, dx_2, dx_3)$ is

$$d\emptyset = \frac{\partial\emptyset}{\partial x_1}dx_1 + \frac{\partial\emptyset}{\partial x_2}dx_2 + \frac{\partial\emptyset}{\partial x_3}dx_3 \qquad (1.59)$$

We can express $d\emptyset$ as a scalar product of two vector s:

$$d\emptyset = \frac{\partial\emptyset}{\partial x_1}dx_1 + \frac{\partial\emptyset}{\partial x_2}dx_2 + \frac{\partial\emptyset}{\partial x_3}dx_3 = (\Delta\emptyset).dr, \quad (1.60)$$

Where

$$\Delta\emptyset \equiv \frac{\partial\emptyset}{\partial x_1}\hat{e}_1 + \frac{\partial\emptyset}{\partial x_2}\hat{e}_2 + \frac{\partial\emptyset}{\partial x_3}\hat{e}_3 \quad (1.61)$$

Is a vector field (or a vector point function).by this we mean to each point $r = \emptyset(x_1, x_2, x_3)$ in space we associate a vector $\Delta\emptyset$ as specified by its three components $(\partial\emptyset/\partial x_1, \partial\emptyset/\partial x_2, \partial\emptyset/\partial x_2)$: $\Delta\emptyset$ is called the gradient of $\emptyset$ and is often written as grad$\emptyset$.

There is a simple geometric interpretation of $\Delta\emptyset$. Note that $\emptyset(x_1, x_2, x_3) = c, c$ being a constant, represents a surface. Let $r = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$be the position vector to a point $P(x_1, x_2, x_3)$ on the surface. If we move along the surface to a nearby point $Q(r + dr)$, then $dr = dx_1\hat{e}_1 + xd_2\hat{e}_2 + dx_3\hat{e}_3$lies in the tangent plane to the surface at P. But as long as we move along the surface $\emptyset$ has a constant value and $d\emptyset = 0$. Consequently from (1.41),

$$dr.\nabla\emptyset = 0 \quad (1.62)$$

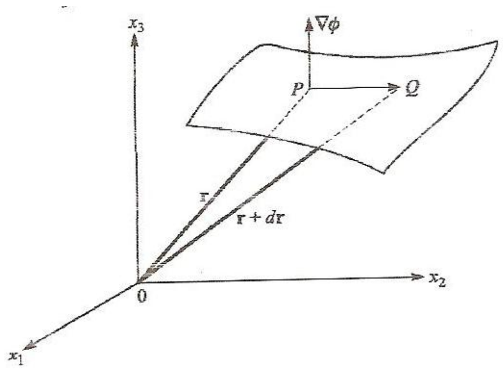


Figure 1.13 gradient of a scalar

Eq. (1.44) states that $\nabla\emptyset$ is perpendicular to dr and therefore to the surface (Fig. 1.12). let us return to

$$\nabla\emptyset = (\nabla\emptyset).dr. \quad (1.63)$$

The vector $\nabla\emptyset$ is fixed at any point P, so that d$\emptyset$ will be a maximum when dr is parallel to $\nabla\emptyset$, since dr. $\nabla\emptyset = |dr||\nabla\emptyset|cos\theta$,and$cos\theta$ is a maximum for $\theta = 0$. Thus $\nabla\emptyset$ is in the direction of maximum increase of $\emptyset(x_1,x_2,x_3)$. The component of$\nabla\emptyset.\hat{u}$ and is called the directional derivative of $\emptyset$ in the direction $\hat{u}$. physically,

this is the rate of change of $\emptyset$ at (x_1,x_2,x_3) in the direction $\hat{u}$.

1.6.7 Conservative vector field

By definition, a vector field is said to be conservative if the line integral of the vector along any closed path vanishes. Thus, if F is a conservative vector field (say, a conservative force field in mechanics), then

$$\oint F.ds = 0 \tag{1.64}$$

$ds$is an clement of the path. A necessary and sufficient condition for the force, F to be a conservative force is that F can be expressed as the gradient of a scalar, say $\emptyset$: $F = -\text{grad}\emptyset$:

$$\int_a^b F.ds = -\int_a^b grad\emptyset.ds = -\int^b d\emptyset = \emptyset(a) - \emptyset(b): \quad (165)$$

It is obvious that the line integral depends solely on the value of the scalar $\emptyset$ at the initial and final points, and $\oint F.ds = -\oint grad\emptyset.ds = 0$.

1.6.8 The vector differential operator∇

We denoted the operation that changes a scalar field to a vector field in Eq. (1.43) by the symbol ∇ (del or nabla):

$$\nabla\equiv \frac{\partial}{\partial x_1}\hat{e}_1 + \frac{\partial}{\partial x_2}\hat{e}_2 + \frac{\partial}{\partial x_3}\hat{e}_3, \tag{1.66}$$

Which is called a gradient operator? We often write $\nabla\emptyset$ as grad$\emptyset$, and the vector field $\nabla\emptyset(r)$ is called the gradient of the scalar field$\emptyset(r)$. Notice that the operator ∇ contains both partial differential operators and a direction: it is a vector differential operator. This important operator possesses properties analogous to those of ordinary vectors. It will help us in the future to keep in mind that ∇ acts both as a differential operator and as a vector.

1.6.9 Vector differentiation of a vector field

Vector differential operations on vector fields are more complicated because of the vector nature of both the operator and the field on which it operates. As we know as in the previous treatment of mathematical implication of vectors, there are two types of products involving two vectors, namely the scalar and vector products; vector differential operations on vector fields can also be separated into two types called the curl and the divergence.

1.6.10 The divergence of a vector

If $V(x_1, x_2, x_3) = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$. Is a differentiable vector field (that is, it is defined and differentiable at each point(x_1, x_2, x_3). In a certain region of space, the divergence of V, written $\nabla.V$ or div V, is defined by the scalar product

$$\nabla.V = \left(\frac{\partial}{\partial x_1}\hat{e}_1 + \frac{\partial}{\partial x_2}\hat{e}_2 + \frac{\partial}{\partial x_3}\hat{e}_3\right).\left(V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3\right)$$

$$= \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}, \qquad (1.67)$$

The result is a scalar field. Note the analogy with $A.B = A_1B_1 + A_2B_2 + A_3B_3$, but also note that $\nabla.V \neq V.\nabla$(bear in mind that ∇ is an operator).$V.\nabla$is a scalar differential operator:

$$V.\nabla = V_1\frac{\partial}{\partial x_1} + V_2\frac{\partial}{\partial x_2} + V_3\frac{\partial}{\partial x_3} \qquad (1.68)$$

What is the physical significance of the divergence? Or why do we call the scalar product $\nabla.V$ the divergence of V? To answer these questions, we consider, as an example, the steady motion of a fluid of density $p(x_1, x_2, x_3)$, and the velocity fieldisgiven by$v(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)e_1 + v_2(x_1, x_2, x_3)e_2 + v_3(x_1, x_2, x_3)e_3$. We now concentrate on the flow passing through a small parallelepiped $ABCDEFGH$ of dimensions $dx_1dx_2dx_3$ (Fig. 1.12). The $x_1$and $x_3$components of the velocity v contribute nothing to the flow through the face$ABCD$. The mass of fluid entering $ABCD$ per unit time is given by $pv_2dx_1dx_3$ and the amount leaving face EFGH per unit time is

$$\left[pv_2 + \frac{\partial(pv_2)}{\partial x_2}dx_2\right]dx_1dx_3 \qquad (1.69)$$

So the loss of mass per unit time is $[\partial(pv_2)|\partial x_2]dx_1dx_2dx_3$.adding the net rate of flow out all three pairs of surfaces of our parallelepiped, the total mass loss per unit time is

$$\left[\frac{\partial}{\partial x_1}(pv_1) + \frac{\partial}{\partial x_2}(pv_2) + \frac{\partial}{\partial x_3}(pv_3)\right]dx_1dx_2dx_3 = (pv)dx_1dx_2dx_3 \quad (1.70)$$

So the mass loss per unit time per unit volume is$\nabla.(pv)$. Hence the name divergence.

The divergence of any vector V is defined as$\nabla.V$. We now calculate $\nabla.(fV)$, wherr f is a scalar:

$$\nabla.(fV) = \frac{\partial}{\partial x_1}(fv_1) + \frac{\partial}{\partial x_2}(fv_2) + \frac{\partial}{\partial x_2}(fv_3)$$

$$= f\left(\frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}\right) + \left(V_1\frac{\partial f}{\partial x_1} + V_2\frac{\partial f}{\partial x_2} + V_3\frac{\partial f}{\partial x_3}\right)$$

Or

$$\nabla.(fV) = f\nabla.V + V.\nabla f \qquad (1.71)$$

It is easy to remember this result if we remember that ∇ acts both as a differential operator and a vector, thus, when operating on fV, we first keep f fixed and let ∇

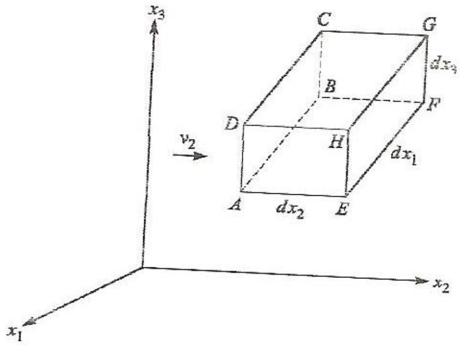


Figure 1.14 Steady flow of a fluid.

Operate on V, and then we keep V fixed and let ∇ operate on $f(\nabla.f$ is nonsense),

And as ∇f and V are vectors we complete their multiplication by taking their dot product.

A vector V is said to be solenoidal if its divergence is zero: $\nabla.V = 0$

1.6.11 The operator ∇^2, the Laplacian

The divergence of a vector field is defined by the scalar product of the operator ∇ with the vector field. What is the scalar product of ∇ with itself?

$$\nabla^2 = \nabla.\nabla = \left(\frac{\partial}{\partial x_1}\hat{e}_1 + \frac{\partial}{\partial x_2}\hat{e}_2 + \frac{\partial}{\partial x_3}\hat{e}_3\right).(V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3)$$

$$= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2},$$

This important quantity

$$\nabla^2 \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \qquad (1.72)$$

Is a scalar differential operator which is called the Laplician, after a French mathematician of the eighteenth century named Laplace? Now, what is the divergence of a gradient?

Since the Laplacian is a scalar differential operator, it does not change the vector character of the field on which it operates. Thus $\nabla^2 \emptyset(r)$ is a scalar field if $\emptyset(r)$ is a scalar field, and $\nabla^2[\nabla\emptyset(r)]$ is a vector field because the gradient $\nabla\emptyset(r)$ is a vector field.

The equation $\nabla^2\emptyset = 0$ is called Laplace's equation.

1.6.12 The Curl of Vector

If $V(x_1, x_2, x_3)$ is a differentiable vector field, then the curl or rotation of V, written $\nabla \times V$ (or curl V or tot V), is defined by the vector product

$$curl\, V = \nabla \times V \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{e}_1 \left(\frac{\partial V_3}{\partial x_2}\frac{\partial V_2}{\partial x_3}\right) + \hat{e}_2 \left(\frac{\partial V_1}{\partial x_3}\frac{\partial V_3}{\partial x_1}\right) + \hat{e}_3 \left(\frac{\partial V_2}{\partial x_1}\frac{\partial V_1}{\partial x_2}\right)$$

$$= \sum_{i,j,k} \varepsilon_{ijk}\, \hat{e}_1 \frac{\partial V_k}{\partial x_j} \qquad (1.73)$$

The result is a vector field. In the expansion of the determinant the operators $\partial/\partial x_1$ must precedeV_i;$\sum_{i,jk}$ stands for $\sum_i \sum_j \sum_j$: and ε_{ijk} are the permutation symbol: an even permutation of ijk will not change the value of the resulting permutation symbol, but an odd permutation gives an opposite sign. That is,

$\varepsilon_{ijk} = \varepsilon_{ijk} = \varepsilon_{ijk} = -\varepsilon_{ijk} = -\varepsilon_{ijk} = -\varepsilon_{ijk}$,and

$\varepsilon_{ijk} = 0$if two or more indices are equal.

A vector V is said to be irrational if its curl is zero:$\nabla \times V(r) = 0$. From this definition we see that the gradient of any scalar field $\emptyset(r)$ is irrigational. The proof is simple:

$$\nabla \times (\nabla\emptyset) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{vmatrix} \emptyset(x_1, x_2, x_3) = 0 \qquad (1.74)$$

becausethere are two identical rows in the determinant. Or, in terms of the permutation symbols, we can write $\nabla \times (\nabla\emptyset)$ as

$$\nabla \times (\nabla \emptyset) = \sum_{ijk} \varepsilon_{ijk}\, \hat{e}_1 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \emptyset(x_1, x_2, x_3).$$

Now ε_{ijk} is ant symmetric in j, k, but $\partial^2/\partial x_j \partial x_k$

Is symmetric, hence each term in the sum is always cancelled by another term:

$$\varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} + \varepsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} = 0, \quad (1.75)$$

And consequently

$\nabla \times (\nabla \emptyset) = 0.$

Thus, for a conservative vector field F, we have

$\text{curl} F = curl\ (\text{grad}\emptyset) = 0.$

We learned above that a vector V is solenoid (or divergence-free) if its divergence is zero. From this we see that the curl of any vector field $V(r)$ must be solenoid:

$$\nabla . (\nabla \times V) = \sum_i \frac{\partial}{\partial x_i} (\nabla \times V)_i = \sum_i \frac{\partial}{\partial x_i} \left(\sum_{j,k} \varepsilon_{ijk} \frac{\partial}{\partial x_i} V_k \right) = 0, \quad (1.76)$$

Because ε_{ijk} is anti symmetric in i, j.

If $\emptyset(r)$ is a scalar field and $V(r)$ is a vector field, then

$$\nabla \times (\emptyset V) = \emptyset(\nabla \times V) + (\nabla \emptyset) \times V. \quad (1.77)$$

We first write

$$\nabla \times (\emptyset \mathrm{V}) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \emptyset V_1 & \emptyset V_2 & \emptyset V_3 \end{vmatrix},$$

$$= \frac{\partial}{\partial x_1} (\emptyset \mathrm{V}_2) = \emptyset \frac{\partial V_2}{\partial x_1} + \frac{\partial \emptyset}{\partial x_1} V_2 \quad (1.78a)$$

So we can expand the determinant in the above equation as a sum of two determinants:

$$\nabla \times (\emptyset V) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ V_1 & V_2 & V_3 \end{vmatrix} + \begin{vmatrix} \hat{e}_2 & \hat{e}_2 & \hat{e}_2 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \emptyset(\nabla \times V) + (\nabla\emptyset) \times V. \qquad (1.78b)$$

Alternatively, we can simplify the proof with the help of the permutation symbolsε_{ijk}

$$\nabla \times (\emptyset V) = \sum_{i,j,k} \varepsilon_{ijk}\, \hat{e}_1 \frac{\partial}{\partial x_1}(\emptyset V_k)$$

$$= \emptyset \sum_{i,j,k} \varepsilon_{ijk}\, \hat{e}_1 \frac{\partial V_k}{\partial x_1} + \sum_{i,j,k} \varepsilon_{ijk}\, \hat{e}_1 \frac{\partial \emptyset}{\partial x_j} V_k$$

$$= \emptyset(\nabla \times V) + (\nabla\emptyset) \times V. \quad (1.79)$$

A vector field that has non-vanishing curl is called a vortex field, and the curl of the field vector is a measure of the vortices of the vector field.

The physical significance of the curl of a vector is not quite as transparent as that of the divergence. The following example from fluid flow will help us to develop a better feeling. Fig 1.15 shows that as the component v_2 of the velocity v of the fluid increases with x_3, the fluid curls about the x_1-axis in a negative sense (rule of the right-hand screw), where $\partial v_2/\partial v_3$ is considered positive. Similarly, a positive curling about the x_1 −axis would result from $v_3$if $\partial v_3/\partial v_2$ we positive. Therefore, the total x_1 component of the curl of v is

$[\mathrm{curl} v]_1 = \partial v_3/(\partial v_2 - \partial v_2/\partial x_3$ (1.80)

Which is the same as the x_1 component of Eq. (1.77).

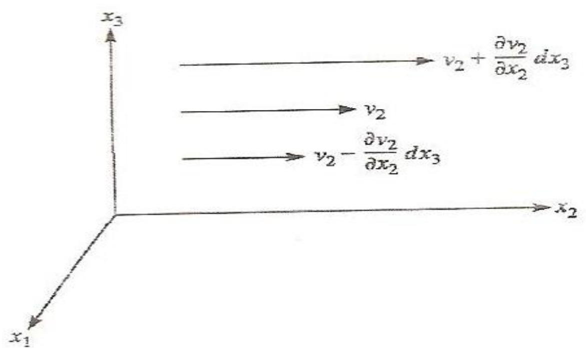


Figure 1.15 Curl of a fluid flow

1.6.13 Formulas involving ∇

We now list some important formulas involving the vector differential operator ∇, some of which are recapitulation. In these formulas, A and B are differentiable vector field functions, and f and g are differentiable scalar field functions of position (x_1, x_2, x_3):

1. $\nabla(fg) = f\nabla g + g\nabla f$;
2. $\nabla.(fA) = f\nabla.A + \nabla f.A$;
3. $\nabla \times (fA) = f\nabla \times A + \nabla f \times A$;
4. $\nabla \times (\nabla f) = 0$;
5. $\nabla.(\nabla \times A) = 0$;
6. $\nabla.(A \times B) = (\nabla \times A).B - (\nabla \times B) \times A$;
7. $\nabla \times (A \times B) = (B.\nabla)A - B(\nabla.A) + A(\nabla.B) - (A.\nabla)B$
8. $\nabla \times (\nabla \times A) = \nabla(\nabla.A) - \nabla^2 A$;
9. $\nabla(A.B) = A \times (\nabla \times B) + B(\nabla \times A) + (A.\nabla)B + (B.\nabla)A$;
10. $(A.\nabla)r = A$;
11. $\nabla.r = 3$;
12. $\nabla \times r = 0$;
13. $\nabla.(r^{-3}r) = 0$;
14. $dF = (dr.\nabla F + \frac{\partial F}{\partial t}dt$ (F a differentiable vector field quantity);
15. $d\varphi = dr.\nabla\varphi + \frac{\partial \varphi}{\partial t}dt$ (φa differentiable scalar field quantity).

1.6.14 Orthogonal curvilinear coordinates

Up to this point all calculations have been performed in rectangular Cartesian coordinates. May calculations in physics can be greatly simplified by using instead of the familiar rectangular Cartesian coordinate system, another kind of system which takes advantage of the relations of symmetry involved in the particular problem under consideration. For example, if we are dealing with sphere, we will find it expedient to describe the position of a point in sphere by the spherical coordinate$(r, \theta, \emptyset)$. Spherical coordinates are a special case of the orthogonal curvilinear coordinate system in order to obtain expressions for the gradient, divergence, curl, and Laplacian. Let the new coordinates u_1, u_2, u_3 be defined by specifying the Cartesian coordinates (x_1, x_2, x_3) as functions of (u_1, u_2, u_3):

$$x_1 = f(u_1, u_2, u_3), x_2 = g(u_1, u_2, u_3), x_3 = h(u_1, u_2, u_3), \quad (1.81)$$

Where f, g, h are assumed to be continuous, differentiable. A point P(Fig. 1.16) in space can then be defined not only by the rectangular coordinates (x_1, x_2, x_3) but also by curvilinear coordinates (u_1, u_2, u_3).

If u_2 and u_3 are constant as u_1 varies, P(orits position vector r) describes a curve which we call the u_1 coordinate curve. Similarly, we can define the u_2 andu_3 coordinate curves through P. We adopt the convention that the new coordinate system is a right handed system, like the old one. In the new system dr takes the form:

$$dr = \frac{\partial r}{\partial u_1} du_1 + \frac{\partial r}{\partial u_2} du_2 + \frac{\partial r}{\partial u_3} du_3 \qquad (1.82)$$

The vector $\partial r/\partial u_1$ is tangent to the u_1 coordinate curve at P. If $\hat{u}_1$ is a unit vector at P in this direction, the $\hat{u}_1 = \partial r/\partial u_2 |\partial r/\partial u_1|$, so we can write $\partial r/\partial u_1 = h_i \hat{u}_1$ where $h_1 = |\partial r/\partial u_1|$. Similarly we can write $\partial r/\partial u_2 = h_2 \hat{u}_2$ and $\partial r/\partial u_3 = h_3 \hat{u}_3$, where $h_2 = |\partial r/\partial u_2|$ and $h_3 = |\partial r/\partial u_3|$, respectively. Then dr can be written

$$dr = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3 \qquad (1.83)$$

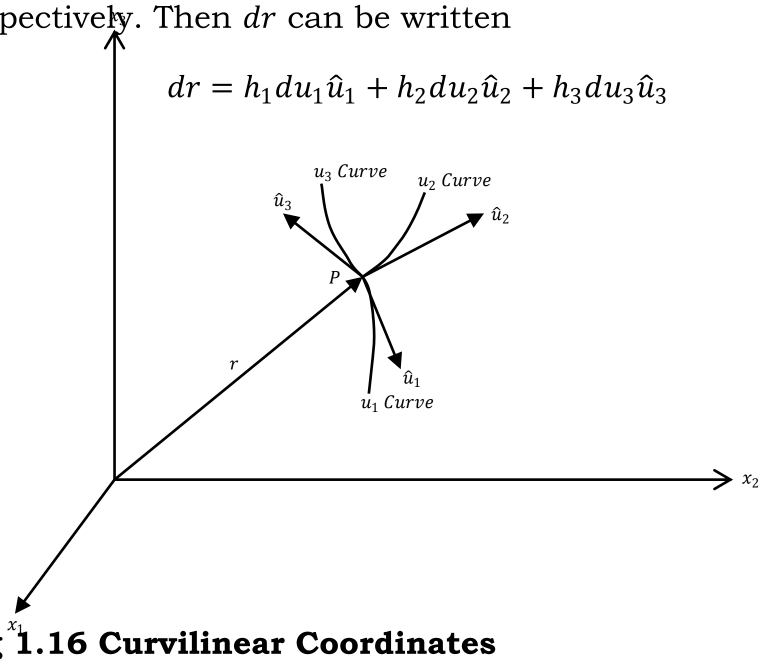


Fig 1.16 Curvilinear Coordinates

Questions

1. What are the direction cosines of a = 12i + 3j – 4k?
2. find the expressions for the angles between a, b, and c where a =3i + 4j + 5k, b=2i-j + 2k, and c=-i+2j-3k
3. Show that p and q are perpendicular where p=3i – 4j + 5k and q= -2i +j 2k.
4. Find the component of the force -3i + 6j + 2k in the direction of the vector -4i + 4j + 7k.
5. If a = 3i – 2j- 4k and b=I +2j – 2k, find a, b and expression for the angle between a and b. if c= 2i –j + k, find c.a and b.c
6. If a=10i – 3j + 5k, b= 2i + bj – 3k, and c= I + 10j -2k, verify that a.b + a.c = a. (b+c).

7. The position vector of a point is given by $r=3t^2, + e^{-t\,j} + 2$ Cos 3tk, where t is time. Find the velocity and allele ratio vectors of the point.
8. Three masses, each of 1kg, have position vectors 2i + 3j, 6i + 4j. And 3i + 2j respectively. Find the position vector of the centre of gravity.
9. Forces f1 = 4i -6j + 5k, F2 = 03i + 2j + 3k, and F3 = 2i + 3j -6k all act on a particle which moved from r1 = 3i + 4j + 5k to r_2 = 2i – 3j + 4k, to r_3 = I + 4j – 2k. Find the work done on the particle in moving from r_1 to r_2, and from r_2 to r_3. what is the work done in moving from r_1 to r_3 ? Neglect gravity forces are given in newtons, distances in meters
10. findaxb, exb and (cxa) xb if a=3i-2j + k, b=-I +j 2k and c=2i + 2j -4k.
11. Verity that a ,bxc and axb .c are equal using the vectors specifically in question (1). (12) show that the rectors a, b and c are coplanar, where a=3i + j -2k, b=-5i +j-2k and c= - i - j – k
12. Find x if the vectors a,b and c are to be coplanar. A=3i -2j + 5k, b=2i –j + k and c=-x i + 2j – k.

CHAPTER 2

2.1.0 Matrix Algebra and Element of Tensors

This section we study the basic algebra and manipulations of matrices and tensors as an example of operators on vector space since vectors can be described in term of their components with respect to the basis. These components may be displayed as an array of numbers called Matrix. The matrix maybe defined as array of scalar real or complex numbers. It is denoted by capital letter.

Example of matrices A, B, C can be written as

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 3 & 3 \\ -2 & -2 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ -3 & 0 \end{bmatrix}$$

$$C = [1 \;\; 2 \;\; 4], \quad C = \begin{bmatrix} 3 & 2 \\ -3 & 4 \end{bmatrix}$$

These numbers inside the bracket are called Matrix element. We note also that the element may contain complex number as given below

$$E = \begin{bmatrix} 1 & -2i & 1 & 4 \\ 2i & 2 & i-1 & 10i \\ 2+3i & -i+1 & 4 & 6 \\ -3 & -3i & 0 & 1+2i \end{bmatrix}$$

A matrix having m row and n column is said to have order m X n. For instant the order of the matrices in examples above are

A: 3X3, B: 3X2, C: 1X3

D: 2X2 and E: 4X4

This implies that a column vector is a matrix having only one column while a row vector is a matrix with only one row.

A matrix with the same number m of column as row is called a square matrix of order m.

Example

$$A = \begin{bmatrix} -2 & 5 & 3 \\ -3 & 0 & 4 \\ 2 & 4 & -1 \end{bmatrix}$$

is a square matrix of order 3.

Each scalar in a matrix is called an element of the matrix. The element of a matrix A is i^{th} row and J^{th} column is denoted by aij

A matrix may be symmetric or anti-symmetric. It is said to be symmetric if

$aji = aij \ \ \forall i \neq j(2.1)$

but anti-symmetric if

$aji = \ -aij \ \forall i \neq j(2.2)$

Examples are:

$$(a)i \quad A = \begin{bmatrix} 1 & -14 \\ -1 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix} (b) \quad B = \begin{bmatrix} 1 & 2 & -3 \\ -2 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

A is symmetric while B is anti-symmetric.

Definition,

A matrix A is said to be diagonal if

$aji = 0 \ \ \forall i \neq j(2.3)$

Such

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2.1.1 Matrix Operation

Normal mathematical algebraic operations can be carried on matrix such as addition, subtraction and multiplication.

Addition and Subtraction

If A and B of the same order mXn, then the matrix C of order mXn defined by

$$Cij = aij + bij \ \ \forall i \neq i, j(2.4)$$

is known as the sum or addition of matrix A and B denoted by A + B

Example

$$A = \begin{bmatrix} 2 & 4 & -14 \\ -71 & 8 & 3 \\ 5 & 2 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 04 & 6 & 3 \\ 27 & -15 \\ 42 & -31 \end{bmatrix}$$

We note that operation of addition of matrices is commutative. That has the following properties

$$A + B = B + A \qquad (2.5)$$

$$(A + B) + C = A + (B + C). \qquad (2.6)$$

In the case of subtraction, if A and B are matrices of the same order mxn, the matrix C of order mxn defined by

$Cij = aij - bij \ \ \forall ij(2.7)$

which is called the difference of A and B is denoted by A-B=C

Example

$$A = \begin{bmatrix} 2 & 4{-}14 \\ -71 & 8 & 3 \\ 5 & 2 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 04 & 6 & 3 \\ 27{-}15 \\ 42{-}31 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 8 & -7 & 3 \\ -5{-}6 & 9 & -2 \\ -1 & 4 & 3 & 0 \end{bmatrix}$$

Here we note that subtraction is not commutative ie

$A - B \neq B - A(2.8)$

2.1.2 Multiplication of Matrix

If A is an mxι matrix and B is ιxn matrix then the matrix C of order mxn defined by

$Cij = \sum_{K=\iota}^{\iota} ai_k L_{kj} \quad - \quad \iota \leq m, j \leq n(2.9)$

Is called the product of A and B and is denoted by AB

Example

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 3 \\ -54 \end{bmatrix}, \quad B = \begin{bmatrix} 276 \\ 103 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 \ 2121 \\ 7 \ 1421 \\ 143542 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 5057 \\ 1813 \end{bmatrix}$$

From this operation, we observe that multiplication of matrices is commutative but has the following properties

$$AB \neq BA$$

$$A(BC) = (AB)C$$

$$(A + B)C = AC + BC$$

$$A(B + C) = AB + AC$$

2.1.3 Null and Identity Matrices

The null or zero matrixes O has all elements equal zero. The properties are

$$(a)\quad AO = A, \quad (b)\quad A + O = A$$

Identity: The matrix I of order nxn defined by

$\delta ij = \delta jk (2.10)$

is called the identity or unit matrix of order nxn (δjk is the Kronicker delta) which is such that for every nXn matrix A

it is often denoted by

$$I_n \begin{bmatrix} 1 & & & & & 0 \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ 0 & & & & & 1 \end{bmatrix} \qquad (2.11)$$

If Aa matrix; then A is referred as the inverse. It is such that

$$A^1 A = AA^1 = I$$

The operation of the inversion of matrices has the following property

$$(AB)^{-1} = B^{-1}A^{-1}$$

2.1.4 Transpose of a Matrix

A matrix A^7 form by interchanging the rows and column of a matrix A is known as the transpose.

For example given a matrix

$$A = \begin{bmatrix} 312 \\ 041 \end{bmatrix}$$

$$A^7 = \begin{bmatrix} 30 \\ 14 \\ 21 \end{bmatrix}$$

forms the transpose of A.

if A is 3X3 order, say

$$A = \begin{bmatrix} 2 & 1 & 4 & -1 \\ 6 & 3 & -3 & -5 \\ 7 & -6 & 4 & 2 \end{bmatrix}$$

Then

$$A^{\mathrm{T}} = \begin{bmatrix} 2 & 6 & 7 \\ 1 & 3 & -6 \\ 4 & -3 & 4 \\ -1 & 5 & 2 \end{bmatrix}$$

is the transpose of A.

The operation of transposition has the following property

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$$

$$(A^{\mathrm{T}})^{\mathrm{T}} = A$$

Hermitian conjugate and Transpose

If $A = aij$ is an mXn matrix then the mXn matrix whose elements are given by $\overline{aij}\ \forall ij$ is called the conjugate of A and is denoted by A.

For example the complex conjugate of the matrix

$$A = \begin{bmatrix} 1 & 23i \\ 1+i1 & 0 \end{bmatrix}$$

is given as

$$A = \begin{bmatrix} 1 & 23i \\ 1-i1 & 0 \end{bmatrix}$$

If A is a matrix then the matrix A^{+} defined by $A^{+} = \overline{A^{T}}$

$$A = \begin{bmatrix} 0 & -5+5i & -10i \\ 0 & -5 & 10 \\ 2i & -1-i & 2 \end{bmatrix}$$

$$A^{+} = \begin{bmatrix} 0 & 0 & 2i \\ -5+5i & -5 & -1+i \\ 10i & 10 & 2 \end{bmatrix}$$

2.1.4 Determinant of a Matrix

If A is 2x2 matrix then the scalar $a_{11}a_{22} - a_{12} - a_{21}$ is known as the determinant which is denoted by $|A|$

For 3x3 matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} (2.12)$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

In this case, it is obtained by deleting the L^{th} row and J^{th} column of a 3x3 matrix. Here what is left is 2x2 matrix whose determinant is called the minor of aij denoted byMij. The scalar $(-1)^{i+j}Mij$ is called the cofactor of aij and is denoted by Aij.

Example

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}$$

$$|A| = 2(2-2) - 3(2+2) + [-1(1+1)] = 3(0) - 3(4) - 1(2) = -12 - 4$$

$$|A| = -16$$

Example

$$A = \begin{bmatrix} 1 & 0 & 5i \\ -2i & 2 & 0 \\ 1 & 1+i & 0 \end{bmatrix}$$

$$|A| = 10$$

If A is a matrix obtained by replacing each element of a given matrix A by its cofactor Aij and then transposing it, is called the ad joint A and is denoted by $\hat{A}$

Example

Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{bmatrix}$$

First, we determined the determinant

$$|A| = 2[-2(2)—2)3 + 4[(-2)(-3) - (1)(2)] + 3[(1)(3)—2)(-3)$$

$$= 11$$

The second step is to determine the cofactor C

$$C = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 13 & -18 \\ -2 & 7 & -8 \end{bmatrix}$$

Third, we find the transpose of C

$$C^T = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{bmatrix}$$

Hence the inverse, A^{-1}

$$A^{-1} = \frac{c^T}{|A|} = \frac{1}{11}\begin{bmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{bmatrix}$$

2.1.5 The rank of a matrix

This is the largest possible square submatrices in which case the determinant may be evaluated.

Example

Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$$

This matrix possesses four submatrices of which the determinants is given as

$$\begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 4 & 1 & 3 \end{vmatrix} = 0 \quad \begin{vmatrix} 1 & 1 & -2 \\ 2 & 0 & 2 \\ 4 & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 2 \\ 4 & 3 & 1 \end{vmatrix} = 0 \quad \begin{vmatrix} 1 & 0 & -2 \\ 0 & 2 & 2 \\ 1 & 3 & 1 \end{vmatrix} = 0$$

Since all these submatrice has zero determinants. The next largest square submatrices obtainable here are 2X2. Considering 2X2 submatrix formed by ignoring third row and the fourth columns of A, we have

$$\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 1\text{X} - 0 - 2\text{X}1 = -2$$

Now since the determinant of this 2X2 matrix is non-zero, the matrix A is of rank 2.

2.1.6 Systems of linear equation

By definition a system of m linear equation in $n-$ unknown $x_1 x_2 x_3 \ldots \ldots . x_m$ is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots & \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \qquad (2.13)$$

Consequently, it may be written in terms of matrices as

$$AX = B$$

Where A is the nxn square matrix given by

$$A = \begin{bmatrix} a_{11}\, a_{12} \cdots .\, a_{in} \\ a_{21}\, a_{22} \cdots .\, a_{2n} \\ a_{n1}\, a_{n2} \cdots .\, a_{nn} \end{bmatrix} c \qquad (2.14)$$

While X and B are the column vectors given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad (2.15)$$

It therefore implies that since the matrix A is invertible with inverse A^{-1} then multiplying it with the above matrix equation gives the solution given as

$$X = A^{-1}B \qquad (2.16)$$

Example

solve for x_1, x_2, x_3.

$$x_1 + 2x_2 - x_3 = 2$$

$$-x_1 - x_2 + x_1 = 1$$

$$x_1 + x_2 = 5$$

$$x_1 + 4x_2 - x_3 + 5x_4 = 0$$

Here

$$A = \begin{array}{cccc} 1 & 2 & 0 & 1 \\ -1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 0' \\ 1 & 4 & -15 & \end{array} \quad A^{-1} = \begin{bmatrix} 15 & -3 & -7 & -3 \\ -10 & 2 & 5 & 2 \\ 5 & 0 & -2 & -1 \\ 6 & -1 & -3 & -1 \end{bmatrix}$$

$$b = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} -8 \\ 7 \\ 0 \\ -4 \end{bmatrix}$$

Thus $x_1 = -8, x_2 = 7, x_3 = 0, x_4 = -4$

Example solve

$$2x_1 + 4x_2 + 3x_3 = 4$$

$$x_1 - 2x_2 - 2x_3 = 0$$

$$-3x_1 + 3x_2 + 2x_3 = -7$$

We represent it by a matrix

$$AX = B$$

$$\begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -7 \end{pmatrix}$$

Obtain A^{-1} and write

$$X = A^{-1}B \text{ which gives}$$

$x_1 = 2, x_2 = -3, x_3 = 4.$

Alternatively, we can use the Crammer's rule to solve the problem. In this case we obtain the determinant of $A, |A|$ and obtain the three Crammer's determinant as

$$\Delta_1 = \begin{vmatrix} 4 & 4 & 3 \\ 0 & -2 & -2 \\ -1 & 3 & 2 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} 2 & 4 & 3 \\ 1 & 0 & -2 \\ -3 & -7 & 2 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} 2 & 4 & 4 \\ 1 & -2 & 0 \\ -3 & 3 & -7 \end{vmatrix}$$

The final solution to the equations then given by

$$x_1 = \frac{\Delta_2}{|A|}, < x_2 = \frac{\Delta_2}{|A|}, x_3 = \frac{\Delta_3}{|A|}$$

$$x_1 = \frac{22}{11} = 2, x_2 = \frac{-33}{11}, x_3 = \frac{44}{11} = 4$$

Eigenvalues and Eigenvector

A vector as we know is a matrix with one row and/ or one column. However, eigenvalues and eigenvector X of a matrix A are those values and vector for which AX = X.

For instance given

$$\underset{A}{\begin{bmatrix} 123 \\ 213 \\ 312 \end{bmatrix}} \underset{X}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \underset{\lambda - X}{6\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

which can easily be written as

$$AX = \lambda IX$$

$$AX - \lambda X = 0$$

$$(A - \lambda)X = 0$$

Where I stands as a unit matrix As the equation represents a set of equations with zero right hand side it has a nontrivial solution if

$$|A - \lambda I| = 0 \qquad (2.17a)$$

In general, it expressed as

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{11} - \lambda \cdots\cdots & & a_{2n} \\ a_{n1} & a_{n2} & \cdots\cdots a_{nn} - \lambda & \end{vmatrix} = 0 \qquad (2.17b)$$

That is theoretically eigenvalue are given by the determinantal equation.

Example:

Find the eigenvalues and eigenvector of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$$

Solution

First of all we write it in terms of characteristic equation

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 100 \\ 010 \\ 001 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - \lambda & 1 & 1 \\ 2 & 3 - \lambda & 4 \\ -1 & -1 & -2 - \lambda \end{bmatrix}$$

$$\begin{vmatrix} 2 - \lambda & 1 & 1 \\ 2 & 3 - \lambda & 4 \\ -1 & -1 & -2 - \lambda \end{vmatrix} = 0$$

$$(\lambda + 1)(\lambda - 1)(3 - \lambda) = 0$$

$$\lambda_1 = 1, \lambda_2 = 1 \; and \; \lambda_3 = 3$$

These are the eigenvalues.

The eigenvector corresponding to each of the eigenvalue can be obtained. For instance the eigenvectors corresponding to eigenvalue λ_2 is given by

$$\begin{bmatrix} 2-1 & 1 & 1 \\ 2 & 3-1 & 4 \\ -1 & -1 & -2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 4 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Hence $x_1 = 1, x_2 = -1, and\; x_3 = 0$

Thus the eigenvector corresponding to $\lambda_2 = 1 \; is \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

We can do the samefor λ_1,λ_3.

Questions

(1) If $A = \begin{bmatrix} 3 & 1 \\ 2 & 3 \\ 6 & 2 \end{bmatrix}$ $and\ B = \begin{bmatrix} 2 & 5 & 3 \\ 1 & 1 & 3 \end{bmatrix}$ show that $AB \neq BA$

(2) Find the inverse of the following matrices (i) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix}$

(3) Find the eigenvalues and the corresponding normalized eigenvector from the following operators (i) $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(4) Use matrix theory to solve the following system of linear equation

(i) $\begin{matrix} x + 2y + 3 = 4 \\ x + 3y + 5 = 2 \\ x\ 5y + 12 = 7 \end{matrix}$ (ii) $\begin{matrix} x_1 + 2x_2 + x_4 = 2 \\ -x_1 - x_2 + x_3 = 1 \\ 2x_1 + x_2 \qquad = 0 \\ x_1 + 4x_2 - x_3 + 5x_4 = 0 \end{matrix}$

2.2.1INNER PRODUCT AND NORMS

The geometry of Euclidean space relies on the familiar properties of length and angle. The abstract concept of a norm on a vector space formalizes the geometrical motion of the length of the length of a vector. The angle between two vectors is governed by their dot product, which is itself formalized by the abstract concept of an inner product.

The most basic example of an inner product is the familiar dot product

$\langle v\ iw \rangle = v.w = v_1w_1 + v_2w_2 + \cdots v_nw_n = \sum_{i=1\ldots}^{n} v_iw_i$, between (column) vectors,$v = (v_1, v_2, \cdots v_n)^T w = w_1, w_2, w_3, \cdots w_n)^T$

Lying in the Euclidean space, $\mathbb{R}^n$. Here note that the dot product of above equation can be identified with the matrix product.

$$v.w = V^T w = (v_1, v_2, v_3 \cdots v_n) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_n \end{bmatrix} \quad (2.18)$$

between a row vector,V^T and a column vector, w.

Now let V be a vector space, real or complete. Then the inner product of $V, W, E\,V$, written as (V, W) as shown is defined as a pairing that takes two vectors $v_1 WEV$ and produces a real

$\langle v\ iw\rangle E\mathbb{R}$. The inner product is refuels to salsify the following three axioms for u, v, WEV,and , $d\ \epsilon\mathbb{R}$.

i. Bilinearity: $\langle cu + dv{:}\, w\rangle = c\langle w\,;w\rangle + d\langle v\,;w\rangle$,

$$\langle w\,;cv + dw\rangle = c\langle r\,;v\rangle + d\langle u\,;w\rangle$$

ii. Symmetry $\langle v\,;w\rangle = \langle w\,;v\rangle$
iii. Positivity $\langle 0\,;0\rangle 0$ whenever $v \neq o$ while $\langle 0\,;0\rangle = 0$.
iv. $\langle cv, w\rangle = c^*\langle v, w\rangle; \langle v, cw\rangle = c\langle v, w\rangle$
v. $\langle v, w + z\rangle = \langle v, w\rangle + \langle v, z\rangle$
vi. $\langle v, w\rangle \leq \|v\|\|w\|$.

Example. Given the vectors a and b in 3 dimensions, i.e.., V_3 we define the inner product as $a, b = a^+ b$

Thus if $a = \begin{bmatrix}1\\0\\1\end{bmatrix}$, and $b = \begin{bmatrix}2\\1\\1\end{bmatrix}$, then $a^+ = [2.19]$

$$\because (a, b) = (a^+ b) = [101]\begin{bmatrix}2\\1\\1\end{bmatrix} = 3$$

In the case of the space of $m \times n$ matrices; The inner product of A and $B\epsilon\, M_{nm}$ is defined as $a, b = Tr\ (A^+ B)$

Where A^+ is the complex conjugate of the transpose of$A$.

e.g. let $A = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$ $and\ B = \begin{bmatrix}0 & -1\\0 & 1\end{bmatrix}$

find the inner product.

$$A^+ = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix};\ A^+ = \begin{bmatrix}-1 & 1\\0 & 1\end{bmatrix}$$

$$(A, B) = \bar{A}^+ B = \begin{bmatrix}-1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} = 1 + 0$$

Practice: Find the inner product of $A, B\ \epsilon\ M_{nm}$ if

$$A = \begin{bmatrix}2 & 1 & 1\\-1 & 3 & 1\end{bmatrix} \text{and} B = \begin{bmatrix}-1 & -1 & 2\\1 & 3 & 1\end{bmatrix}.$$

A vector space equipped with an inner product is called an inner product space. It can be noted that a given vector space can admit many different inner products.

Given an inner product, the associated norm of a vector $v \epsilon V$ is defined as the positive square root of the inner product of the vector with itself:

$$\|v\| = \sqrt{\langle v\,; w\rangle} \qquad (2.19)$$

The positivity axiom implies that $\|v\| \geq 0$ is real and non-negative, and equal 0 if and only if $v = 0$ is the zero vectors.

2.2.2 INNER PRODUCTS OF FUNCTION SPACE

Inner product and norms on function space play some essential role in modern analysis partiality Fourier analysis and the solution to boundary value problems for both ordinary and partial differential equations.

E.g. Given a bounded dosed interval $[a, b]c\ \mathbb{R}$, consider the vector space $C^0 = C^0[a, b]$ consisting of all continuous functions$F: [a, b \rightarrow \mathbb{R}$. The integral

$$\langle f : g\rangle = \int_a^b f(x)\, g(x) dx \qquad (2.20)$$

We define an inner product on the vector space C^0, as we will be seen below. The associated norm is accordingly given as

$$\|f\| \sqrt{\int_a^b f(x)^2\, dx} \qquad (2.21)$$

This quantity is known as the L^2 norm of the function f over the interval$[a, b]$.

For example, if we take $[a, b] = [0, {}^{\pi}/_{2}]$, then the L^2 inner product between $f(x) = \sin x$ and $g\ (x) = cosx$ can be obtain.

$$\langle sinx; cosx\rangle = \int_0^{\pi/2} sinx\ cosx\ dx = \left[\frac{1}{2} sin^2 x\right]_0^{\pi/2} \frac{1}{2}$$

Similarly the norm of the function $sinx$ is

$$\|\sin x\| = \sqrt{\int_0^{\pi/2} (\sin x)^2\, dx} = \sqrt{\pi/4} = \frac{\sqrt{\pi}}{2}$$

We one is dealing with the L^2 inner product or norm, one should always be careful to specify the function space, or equivalently the interval on which it is being evaluated.

2.2.3 NORMS

A norm on the vector space V assigns a real number$\|V\|$ to each vector $v \epsilon V$, subject to the following axioms for all $v, w \epsilon V$, and $c \epsilon \mathbb{R}$.

i. Positivity: $\|V\| \geq 0$, with $\|V\| = 0$ if and only if $V = 0$.
ii. Homogeneity: $\|cv\| = |c|\|v\|$.

It is important to note that every inner product gives rise to a norm that can be used to measure the magnitude or length of the elements of the underlying vector space and that an inner product gives rise to a norm. Though in general not every norm used in analysis and applications arises from an inner product.

iii. $\|v + w\| \leq \|v\| + \|w\| \; \forall \; v, w \epsilon V$ (Triangular inequality). The norm of a vector is its "distance" from the origin.$\|v\|$is the norm of V as already indicated.

In the case where $V = R$, the real number line, the norm is the absolute value $|v|$.

If the norm of v in the vector space V is unity, situation where any vector is not normalized, we can normalize it by dividing by the norm Example given a vector $a = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ normalize the vector.

The norm of a is $\|a\| = \sqrt{(a, a)}$

$$(a, a) = [101] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2$$

$$\|a\| = \sqrt{(a, a)} = \sqrt{2}$$

The normalized vector of a is $C = \frac{a}{\|a\|}$

$$C = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Another example, normalize the vector $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Now, the vector is in form of matrix mxn the norm is defined as

$$\|A\| = \sqrt{T_r(A, A)}$$

$$A^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} and \bar{A}^+ = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\bar{A}^{+}A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T_r(\bar{A}^{+}A) = \|A\| = \sqrt{(2-1)^2 + (1-0)^2} = \sqrt{1+1} = \sqrt{2}$$

The normalized $A = \frac{A}{\|A\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Linear independence and dependence vectors $V_1 \cdots V_k \epsilon V$ are called linearly dependent if there exist a collection of scalars $C_1, \cdots C_k$ not all zero such that

$$C_1V_1 + \cdots C_kV_k = 0$$

Vectors which are not linearly dependent are called linearly independent.

Example given vector $V_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, V_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, V_3 = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$, verify whether they are linearly dependent or not.

From (iii) $C_1 = -C_3$ and from (ii)$C_2 = -2C_3$, putting these in (i) gives

$$-C_3 + 2C_3 + C_3 = 0$$

$$-2C_3 = 0 \; or, C_3 = 0$$

$$\therefore C_1 = C_2 = C_3 = 0$$

This enables one to conclude that the set is linearly independent

1. The set $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ linearly dependent or independent? Normalize each vector.

2.2.4 ORTHOGONALITY AND ORTHORNALITY

If any vector in the vector space V, can be written as a linear combination.

$V = a_1Q_1 + a_2Q_2 + \cdots + a_nQ_n = \sum_{i=1}^{n} a_i\, Q_i$ (2.22)

Then we say the space is spanned by the complete orthonormal basis$\{Q_i\}$.

Vector say v_1 and v_2 in a vector space V are orthogonal if $(V_1.V_2) = 0$

Suppose there exist a linearly independent set

$\{Q_i\}_{i=1}^{n}$ $i.e$ $\{Q_1Q_2.Q_n\}$ such that$\{Q_i, Q_j\} = 0, i \neq j$ implies orthogonality.

If on the other hand, $\{Q_i, Q_j\} = 1$, this is said to be an orthonormal set. It is normally written as

$$\{Q_i, Q_j\} = \delta_{ij} \begin{cases} 0 \; for \; i \neq j \\ 1 \; for \; i \neq j \end{cases}$$

2. Determine whether the vectors

$$V_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, V_2 = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} V_3 = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}, V_4 \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

are linearly independent or linearly dependent.

By inspection, $V_1 = 2V_2 + V_3 = 0$

On the other hand, the first two vectorsv_1, v_2 are linearly independent.

Now, $C_1 V_1 + C_2 V_2 = 0$

$$C_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = 0, \begin{bmatrix} c_1 & + & 0 \\ 2C_1 & + & 3C_2 \\ -C_1 & + & C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 + 0 = 0,$$

$$2C_1 + 3C_2 = 0,$$

$$-C_1 + C_2 = 0,$$

$$C_1 = 0, 2C_1 + 3C_2 = 0, -C_1 + C_2 = 0$$

Which has only the trivial solution $C_1 = C_2 = 0$showing linear independence?

Example 2

Show that the set $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is linearly independent.

Solution

$$C_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From which we obtain

$$C_1 + C_2 + C_3 = 0, C_2 + 2C_3 = 0, C_1 + C_3 = 0$$

2.3.1 ELEMENT OF TENSOR ANALYSIS

Tensors are important in many areas of physics including general relativity and thermodynamics. Scalars and vectors are special case of tensors. A quantity that does not change under rotations of the co-ordinate system in three dimensional spaces, an invariant is known as scalar. A scalar thatis specified by one real number is a tensor of rank zero. A quantity whose components transforms under rotations like those of the distance of a point from a chosen origin is a vector quantity.

Tensors are a natural generalization of vectors. Tensors are defined by means of their properties of transformation under coordinate transformation.

Let us consider the transformation system $(x^1, x^2 \cdots\cdots, x^N)$ to another $(x^1, x^{1\ 2} \cdots\cdots, x^{1N})$ in N −dimensgional spaceVN. We note that in writing x^v is a superscript and should not be mistaken for an exponent. In $3-\partial$ at space we use subscripts. When we transform the coordinates, their differentials transform according to the relationis

$$dx^{\mu} = \frac{\delta x^{\mu}}{\delta x^{1^v}} \quad dx^{1^v}. \quad (2.23)$$

Here we have sued Einstein's summation convention; repeated indexes which appear. Once in the lower and once in the upper position are automatically

summed over. Thus

$$\sum_{N=1}^{N} A_N \ A^N = A_N A^N \ (2.24)$$

It is important to note that indexes repeated in the lower part or upper parts alone are not summed over. An index which is repeated and over whosesummation is repeated is called a dummy index. Clearly by any other index that does not appear in the same term.

A set of N quantities $A^N(N = 1,2,\cdots,N)$ which, under a coordinate change transform like the coordinate differentials, are called the components of a contravariant vector or a contravariant tensor of the first rank or first order.

$$A^u = dx^{\mu} = \frac{\delta x^{\mu}}{\delta x^{1^v}} A^{1^v} \ (2.25)$$

This relation can easily be inverted to express A^{1^v} in terms of A^u.

If Nquantities $A^u(n = 1,2,\cdots,N)$in a coordinate system $(x^1, x^{1\ 2} \cdots\cdots, x^{1N})$ by the transformation equations

$$A_u = dx^{\mu} = \frac{\delta x^{\mu}}{\delta x^{1^v}} A_v \ (2.26)$$

They are called components of a covariate vector or covariant tensor of the first rank or first order.

2.3.2 TENSORS OF SECOND RANK

Vector or first order tensor normally defined as an entity with three components which transformed in a certain faction under rotation of the coordinate frame. Second order Cartesian tensor similarly is defined as an entity has nine components say $A_{ij}, i, j = 1,2,3$ in the certesian frame of reference 0123 which on rotation of the frame of reference to $0\bar{1}\bar{2}\bar{3}$become.

$$A_{pg} = lip\, A_{ij} \quad (2.27)$$

By the orthogonally. Properties direction confinesl_{rs} we have the inverse transformation$A_{ij} = lip\, ljq\, A_{pg} \quad \cdots B_i$

To indicate that a given entity is a second order tensor we have to show that its components transform according the above equation B_1. A valuable means of establishing tensor character is the quotient rule. A second order tensor may be writer down as a 3×3 matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} (2.28)$$

And it is occasionally convenient to treat it as such. In the rotation of matrices the transformation above would be written as $L^1 AL = \bar{A}, or\ A = L\bar{A}L^1$.

from two contravariant vectors A^u and B^v we may form the N^2 quantities $A^u B^v$. This is known as the outer product or of tensors. These N^2 quantities form the components of a contravariant tensor of the second rank: km T^{uv} which under a coordinate change, transform like the product of two contravariant vector.

$$T^{uv} = \frac{dx^u}{dx^\propto} \frac{dx^v}{dx\,1^B} T^1 \propto B_1 \quad (2.29)$$

Is a contra variant tensor of rank two? We may also form a covariant vector, which transforms according to the formula

$$T_{uv} = \frac{dx^u}{dx^\propto} \frac{dx^{v1}B}{dx\,1^B} T^1 \propto B \qquad (2.30)$$

Similarly, we can form a mixed tensor T_v^u of order two that transform as. We may continue the process and multiply more than two vectors together, taking care that their indexes are all different. In this way one can construct tensor of higher rank. The total number of free indexes of a tensor is its rank (or order). In a Cartesian coordinate system, the distinction between contravariant and the covariant tensors vanishes. This can be shown with velocity and gradient vectors. Velocity and acceleration are contravariant vectors, they are represented in terms of components in the direction of coordinate increase; the grader vector is a covariant vector and it is represented in terms of components in the directions. Orthogonal to the constant x^u surface hence the distinction between the covariant and contravariant vectors quantities. In fact this is the essential difference between contravariant tensor; a covariant tensor is represented by components in directions orthogonal to like constant coordinate surface, and a contravariant tensor is represented by component in the direction of increasing coordinate and if the two tensor have the same contravariant rank and of the same covariant rank, way say that they are of the same type.

2.3.3 BASIC RULES OF OPERATION WITH TENSORS

1. Equality; Two tensors are said to be equal if and only if they have the same covariant rank and the said contravariant rank, and every component gone is equal to the corresponding component of the other

$$A_m^{\propto B} B_M^{\propto B}. \qquad (2.31)$$

2. Addition (subtraction). The sum (difference) of two or more tensors of the same type and rank is also a tensor of the same type and rank
3. Addition of tensors is commutative and associative.
4. Outer product of tensors: The product of two tensors is a tensor whose rank is the sum of the ranks of the given two tensors. This product involves ordinary multiplication of two components of the tensor and is called the outer product. For instance. $A_m^{\propto B} B_M^{\propto B} = C_m^{v\propto B}$Is the outer product of $A_m^{\propto B}$and $B_{\curlywedge}^{B}$.
5. Contraction; if a covariant and a contravariant index of a mixed tensor are equal; a summation over the equal indexes is to betaken according to the summation convention. the resulting tensor is a tensor of rank two less than that of the original tensor. This process is called contraction. For instance, if we start with forth order tensor T^m one way of contracting it is $vp,$ to set$d = p$, which gives the second rank tensorT_{mp}^{mp}. One could contract it again to get the scalarT_{mp}^{mp}.
6. Inner product of tensor: The inner product of two tensor is produced by contracting the outer product of the tensors. For example, given two tensors$A_d^{\propto B}$and$B_v^{\alpha\beta}$, the outer product is $A_m^{\propto B} B_M^{\propto B}$ setting δ= μ, we obtain the inner product$A_m^{\propto B} B_v^m$.

Symmetric and antisymmetric tensors a tensor is said to be symmetric with respect to two contravariant or two covariant indices if its component remains unchanged upon interchange of indices.If $A_{ij} =$

A_{ji}. The tensor is said to be symmetric and a symmetric tensor has only six distinct components. If $A_{ij} = A_{ji}$ the tensor is said to antisymanetric and such a tensor is characterized by only three scalar quantities for the diagonal terms A_{11} are zero. The tensor whose $ijth$ element is A_{ji} is called the transpose$A^1 of A$.

The analogyis the same with that of a matrix allows one to define the conjugate second order tensor the determinant of a tensor A is the determinant of the matrix A.

$$\det A = \in_{ijk} \; A_1, \;\; A_{zy} A_{3k} \quad (2.32)$$

If this is not zero we can find the inverse matrix by dividing the cofactor of each elements by the determinant and transposing. This is called the conjugate tensor.

7. For example if we let $x = P_{\curlywedge mv} A^{\curlywedge}$ be an arbitrary contravariant vector and $A^{\curlywedge} P_{\curlywedge mv}$ be a tensor, say Q_{mv}: $A^{\curlywedge} P_{\curlywedge mv} = Q_{mvs}$ then be of the same order to be added; a vector cannot beaded to a second order tensor.

2.3.4 SCALAR MULTIPLICATION AND ADDITION

If $\propto$ is a scalar and A a second order tensor, the scalar product of $\propto$ and A is a tensor $\propto A$ each of whose components is $\propto$ time the corresponding component of A as stated in property number 3.The sum of two second order tensors is a second order tensor each of whose components is the sum of two second order tensors is a second order tensor each of whose components is the sum of the corresponding components of the two tensors.

Thus the $ijth$ component of $A + B$is $A_{ij} + B_{ij}$. We note that tensors most be of the same order to be added; a vector cannot be added to a second order tensor.

A linear combination of tensors results from using both scalar multiplication and addition. $A + BB$is the tensor whose $ijth$ component is $\propto A_{ij} + BB_{ij}$. Subtraction may therefore be defined by putting $\propto = 1 \; B = -1$.

Any tensor maybe represented as the sum of a symmetric part and an ant symmetric part. For $A_{ij} = \frac{1}{2}\left(A_{ij} + A_{ij}\right) + \frac{1}{2}\left(A_{ij} - A_{ij}\right)$ and interchange i and j in the first factor leaves it in changed but changes the sing of the second. Thus

$A = \frac{1}{2}(A + A^1) + \frac{1}{2}(A.A^1)$ .Represents A as the sum of a symmetric tensor and antiemetic tensor.

2.3.5 CONTRACTION AND MULTIPLICATION

The operation of identifying two indices of a tensor and so summing on them is known as contraction as stated in the fourth property. $A_{ii}$is the only contraction of A_{ij}

$$A_{ii} = A_{11} + A_{22} + A_{33} \quad (2.33)$$

Which is no longer a tensor of the second order but a scalar, or tensor of order zero? lip$lip = \delta_{ij}$ by the orthoganality of the lij. The scalar A_{ii} is known as the trace of the second order tensor $ie\ tr\ A$

If A and B are two second order tensors 81 numbers can be formed from the products of the 9 components of each,

$A_{ij} B_{km}, i, j, k, m = 1,2,3$. The full set of these products are components of a fourth order tensor, which is yet to be defined?

2.3.6 Isotropic tensors.

These are tensors whose components are unchanged by rotation of the frame of reference is known as isotropic tensor but the one whose

components changes by rotation of frame of reference are called anisotropic tensor which is the type tensor used the analysis and study of electromagnetic wave polarization in bi-axial crystals.

The trivial case of this is the tensors of all orders whose components are all zero. All tensors of the $zeroth$ order are isotropic and there are no first orders are isotropic tensors. The only isotropic become order tensoris d_{ij}. That $A_{ij} = d_{ij}/r^2$ is an isotropic tensor because it is geometrically invariance under rotation.

We have

$$\hat{g} = (g_{mv}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.34a)$$

$$\hat{g} = (g_{mv}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & r^2 & sin^2\theta \end{pmatrix} \quad (2.34b)$$

In an$N$ −dimensional orthogonal coordinate system $g_{mv} = 0\ for\ m \neq v$ and m a Cartesian coordinate system, $g_{vm} = 1\ and\ g_{mv} = for\ m \neq v$.

In general case of Riemannian space, the g_{mv} are functions of the coordinates $x^{\ell}(m\ 1,2, \cdots N)$.

Since the inner product of g_{mv} and the contravariant tensor $dx^u dx^v$ is a scalar(ds^2, the square of line element), the according to the quotient law g_{mv} is a covariant tensor.

To do this,

$$ds = g_{\propto B} dx^{\propto} dx^{B} = g^{1}_{\propto B} d^{1\propto}_{x} dx^{1B}$$

$$g^{1}_{\propto B} = \frac{\delta x^{1\propto}}{\delta x^{m}} \frac{\delta x^{1B}}{\delta x^{v}} dx^{m} dx^{v} = g_{mv} dx^{m} dx^{v}$$

$$\text{Or } \left(g^{1}_{\propto B} = \frac{\delta x^{1\propto}}{\delta x^{m}} \frac{\delta x^{1B}}{\delta x^{v}} - g_{mv}\right) dx^{m} dx^{v} = 0 \qquad (2.35)$$

The above equation is identically zero for arbitrary dx^{m} , so we have

$$g_{mv} = \frac{\delta x^{1\propto}}{\delta x^{m}} \frac{\delta x^{1B}}{\delta x^{v}} g_{\propto B} \dots A_1 \qquad 2.36)$$

Which shows that g_{mv} is a covnariant tensor rank two. It is called the metric tensor or the fundamental tensor. Now contravariant and covariant tensors can be converted into each other with the help of the metric tensor. For example, we can get the covariant vector (tensor of rank one) A_m from the contravariant vectorA^v:

$$A_m = g_{mv} A^{v} \cdots \quad A_2 \ (2.36)$$

Since we expect that the determinant of g_{mv} does not vanish, the above equations can be solved for A^v in terms of theA_m. Let the result be

$$A^{v} = g^{vm} A_{m}, \cdots, \qquad A_3$$

By combining equation A_2 and A_3 we get $A_m = g_m g^{v\propto} A_{\propto}$

Since the equation must hold for any arbitrary

$$A_m\text{,we hav } g_{mv} g^{v\propto} = \delta^{\propto}_{m} \ (2.39)$$

Where $\delta^{\propto}_{m}$ is Kronecher's Deltasymbol .Thus, g^{mv} is the inverse of g_{mv} and vice versa; g^{uv} is often called the conjugate or reciprocal tensor of g_{mv} are the contravariant and covariant components of the same tensor that is the metric tensor. Notice that the matrix(g^{mv}) is just the inverse of the matrix (g_{mv}).

We can use g_{mv} to lower any upper i9ndex index occurring in a tensor and use g^{mv} to raise any lower index. It is necessary to remember the position from which the index was lowered or raised, because when we bring the index back to its original site, we do not want to interchange the order of indices, in general $T^{mv} \neq T^{vm}$.

Thus, for example

$$A_g^p = g^{1p} A_{rg}, A^{pg} = g^{rp} g^{sq}, A_{rs}^p = g_{rg} A_s^{pq} \quad (2.40)$$

2.3.7ASSOCIATED TENSORS

All tensors obtained from a given tensor by forming an inner product with the metric tensor are called associated tensors of the given tensor. For example $A^{\propto}$ and $A_{\propto}$ are associated tensors

$$A_{\propto} = g_{\propto B} A^B, \ \ A^{\propto} = g^{\propto B} A_B \ 2.41)$$

CHAPTER 3

3. 1.0 Complex Number

Complex number is as a result of some algebraic equations that have none real/rational roots as we shall show below.

In a quadratic equation of the form

$$ax^2 + bx + c = o \qquad (3.1)(a, b, c\ rational)$$

Does not always have rational roots; for instance, the roots of

$$ax^2 + 2x - 1 = 0\ \ are$$

$$-1 \neq \sqrt{2}\ in\ some\ cases$$

If $ax^2 - 4x + 5 = 0$

$$(x-2)^2 = -1$$

$$x - 2 = \neq \sqrt{(-2)} \neq i$$

$$x = 2 \pm i$$

$2 - i, 2 + i$ (3.2)

NBThe equation $ax^2 + bx + c = 0$ have two roots if $h_2 > 4ac$ is clumsy.

1. Express $\frac{a+ib}{c+id}$ in the form $p + iq$

solve$\frac{a+ib}{c+id} \times \frac{(c-id)}{(c-id)} = \frac{ac+icb-iad-be}{C^2+d^2}$

$$= \frac{ca - bd + i(cb - ad)}{C^2 + d^2}$$

$$\frac{ca - bd}{C^2 + d^2} = \frac{i(cb - ad)}{C^2 + d^2}$$

$c + id$has$c - id$ as the conjugate complex numbers.

$E\ G$ Express (1)$\frac{2+3i}{1+i}$ in the form $p + iq$.

(2)$\frac{\cos\theta + i\sin\theta}{\cos\phi - i\sin\phi}$

3.1.2 Argan Diagram

This involves the representation of complex number graphically

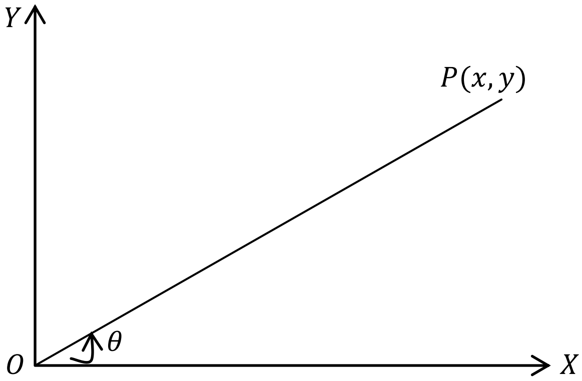
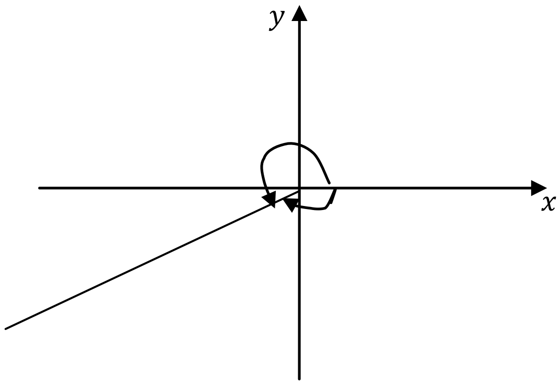


Fig2.1 showing r5epresentation of complex number in Cartesian plane withOP s the radius vector

$\gamma^2 = x^2 + y^2 : \gamma = \sqrt{x^2 + y^2}$ (3.3)

This direction specifying the radius vector OP is not quite so easy to deal with because there are infinitely many positive and negative angle which would do

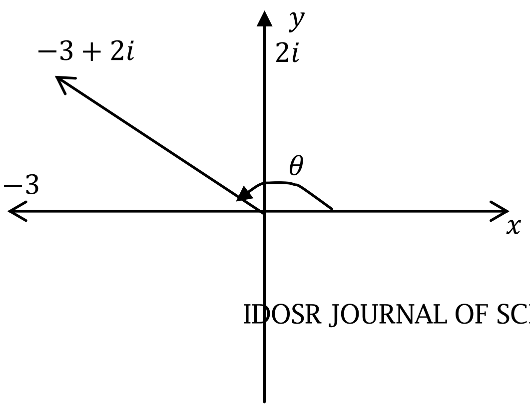


The angle between the radon vector OP and the positive $x - axis$ is called the argument of the complex number $x + iy$.

$Arg(x + iy)$. It has infinitely many values

$$-\pi < arg(x + iy) \leq \pi$$

Eg mark on the Argand diagram the radius vectors corresponding to



$$r = \sqrt{(-3)^2 + 2^2} = \sqrt{9+4}$$
$$r = \sqrt{13}$$
$$\cos\theta \frac{x}{r} = \frac{-3}{\sqrt{13}}$$
$$\theta = cos^{-1}\frac{-3}{\sqrt{13}}$$

Eg

The conjugate of $x + iy$ is $x - iy$ and for brevity we write $z = x + iy, \bar{z} = x - iy$

Show that

$$z\bar{z} = |x+iy|^2 = |x-iy|^2$$

What are the values $= |z\bar{z}|, \left|\frac{z}{\bar{z}}\right|, \arg(z\bar{z})$?

a) $|x+iy| \times |x+iy| = (x^2+y^2)^{1/2}$
$$= x^2 + y^2$$

b) $|x-iy| \times |x-iy| = (x^2+y^2)^{1/2}(x^2+y^2)^{1/2}$
$$= x^2 + y^2$$

$$|z\bar{z}| = |(x+iy)(x+iy)| = x^2 + y^2$$

$$\left|\frac{z}{\bar{z}}\right| = \left|\frac{x+iy}{x-iy}\right| = \frac{x^2+y^2}{x^2+y^2} = 1$$

3.1.3 De Moivre Theorem

The theorem states that for any rational value of n, one value of $(\cos\theta + \sin\theta)^n$ is given by $(\cos\theta + \sin\theta)^n = \cos n\,\theta + \sin n\,\theta$

Prove that

$$(\cos\theta + i\sin\phi)(\cos\phi + i\sin\phi)$$

$$\cos\theta\cos\phi + i\sin\phi + i\sin\ \theta\cos\phi - \sin\theta\sin\phi$$

$$\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \sin\theta\sin\phi)$$

$$\cos(\theta+\phi) + i\sin(\theta+\phi)$$

Therefore if $\theta = \phi$,

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2$$

Express in the form $x + iy$

1. $\frac{\cos\ \theta + i\sin\ \theta}{\cos\ \phi + i\sin\ \phi}$

Find the roots of unity

$$z^z = 1; z^3 - 1 = 0$$

Using De Moivre's theorem, we can express 1 as a complex number in infinitely many ways.

$$\cdots\cdot \cos(-2\pi) + i\sin(-2\pi), \cos o + i\sin o \cos 2\pi + i\sin 2\pi, \cos 24\pi + i\sin 4\pi,$$

Or generally $\cos 2k\pi + i\sin 2k\pi$,

Where is an integer $k = 0,1,2,3,4 \cdots n$.

$$\therefore if z^3 = 1$$

$$\Rightarrow z = \sqrt[3]{1}$$

$$= \sqrt[3]{(\cos\theta + i\sin\theta)}$$

$$= (\cos\theta + i\sin\theta)^{1/3} = \cos\frac{\theta}{3} + i\sin\frac{\theta}{3}$$

$$\therefore \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right), \cos\frac{\theta}{3} + i\sin\frac{\theta}{3}, \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}.$$

Find the square roots of each of the following complex numbers

ii. $3 + 4i$ $24 = 10i$

solution

let$z^2 = 3 + 4i, z = x + iy$

$$\therefore (x + iy)^z 3 + 4i$$

$$x^2 + 2xyi - y^2 = 3 + 4i$$

$$x^2 - y^2 + 2xyi = 3 + 4i$$

$$x^2 - y^2 = 3$$

$$2xyi = 4i$$

$$2xy = 4$$

$$y = 4/2x = 2/x$$

Substituting in (1) in place of y

$$x^2 - \frac{4}{x^2} = 3$$

$$x^4 - 3x^2 - 4 = 0$$

$$(x^2-4)(x^2+1)=0$$

$$x^2+1=0 \Rightarrow x^2+1 \neq 0$$

$$x^2-4=0 \Rightarrow x=\pm 2$$

From $y=2/x, y=1$

$$x=-2,2, y,=-1$$

The square root of $3+4i$ *are*

$$(-2-i),(2+i)$$

Find the real number x and y such that $(x+yi)(2-3)=-13i$

$$x+yi=\frac{-13i}{2-3i}$$

$$=\frac{-13i(2-3i)}{(2-3i)(2-3i)}=\frac{-26i+39}{13}$$

$$x+yi=\frac{-26i}{13}+\frac{39}{13}$$

$$x=3, y=-2$$

1) $(x+yi)(1+4i)=6+7i$
2) $(1+2)x+(2-3i)y=10$
3) $\frac{x}{2+i}+\frac{y}{2+3i}=4+i$

Derivative of complex variables

Find the derivative of $w=z^2$

Using the first principles

$$w=z^2$$

$$w+\partial w=(z+\partial z)^2=z^2+2z\partial z+\partial z^2$$

$$\partial w=2z\partial z+\partial z$$

$$\frac{\partial w}{\partial z}=2z+\partial z$$

$$\frac{dw}{dz}\lim_{\partial \to 0}(2z+\partial z)=2z$$

2. Find the derivative of $w=z\bar{z}$ where

$$z = x + yi \quad \bar{z} = x - yi$$

Solution

$$w = z\bar{z} \therefore w + \partial w = (z + \partial z)(\bar{z} + \partial \bar{z})$$

$$w + \partial w = z\bar{z} + z\partial\,\bar{z} + \partial\bar{z}\partial z + \partial z\partial z$$

$$\partial w = z\partial\,\bar{z} + \bar{z}\partial z + \partial z\partial\bar{z}$$

$$\frac{\partial w}{\partial z} = \bar{z} + \frac{z\partial\,\bar{z}}{\partial z} + \partial\bar{z}$$

$$\frac{\partial w}{\partial z} - (x - iy) + (x + yi)\left[\frac{\partial x - i\partial y}{\partial x + i\partial y}\right] + \partial x - i\partial y$$

For $\left.\begin{matrix} z = x + yi \therefore \partial z = \partial x + i\partial y \\ \bar{z} = x - yi \therefore \partial\bar{z} = \partial x - i\partial y \end{matrix}\right\}$

$$\frac{\partial\bar{z}}{\partial z} = \frac{\partial x - i\partial y}{\partial x + i\partial y}$$

1) 1^{st} let $\partial y \to 0; \frac{\partial w}{\partial z} = x - iy + (x + iy)\frac{\partial x}{\partial x} + \partial x$

   Then $\frac{\partial w}{\partial z} = \lim_{\partial x > 0}[x - iy + x + iy + \partial x] = 2x \qquad (A)$

2) $\frac{\partial w}{\partial z} = (x - iy) + (x + iy)\left[\frac{\partial x - i\partial y}{\partial x + i\partial y}\right] + \partial x - +\partial y$

If $\partial x \Rightarrow 0$

$$\frac{\partial w}{\partial z} - (x - iy) + (x + yi)(-1) - i\partial y = -i2y - i\partial y$$

Then $\frac{\partial w}{\partial z} = \lim_{\partial y => 0}[-i2y - i\partial y] = -i2y$

$$\frac{\partial w}{\partial z} = -i2y \qquad (B)$$

The two results A and B are clearly not the same of all values of x and $y -$ with one exception, ie

When $x = y = 0$

Therefore $wz\bar{z}$ is a function that has no specific derivative, except at $z = 0 -$ and there are other. If would be convenient, therefore, to have some form of test to see whether a particular function $w = f(z)$has a derivative $f'(z)$ at $z = z_0$.This useful tool is provided by the $Cauchy -$ Riemann equations.

If $w = f(z) = U + i\,V$,we have to establish conditions for $w = f(z)$ to have a derivative at a given point $z = z_0$

$$w = U + j\,V \therefore \partial w = \partial\, u + i\partial\, V;$$

$$z = x + iy \therefore \partial\, z = \partial x + i\partial\, y$$

Then $f'(z) = \frac{\partial\, w}{\partial z} = \lim_{\partial z \to 0} \frac{\partial\, u + i\partial\, V}{\partial\, z} = \lim_{\substack{\partial x \to 0 \\ \partial y \to 0}} \left[\frac{\partial\, u + i\partial\, V}{\partial\, x + i\partial\, y}\right]$

(a) let $\partial x \to 0$, followed by $\partial y \to 0$

Then from (1) above, $f'^{(z)} = \frac{\partial\, w}{\partial z} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

For $f'(z) = \lim_{\partial y \to 0} \left[\frac{\partial\, u + i\partial\, V}{i\partial\, y}\right] = \lim_{\partial y \to 0} \left[\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right]$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

(b) let $\delta y \to 0$, followed by $\delta x \to 0$

$$\frac{\partial\, w}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad (3.4)$$

for $f'(z) = \lim_{\partial x \to 0} \left[\frac{\partial\, u + i\partial\, V}{\partial\, x}\right] = \lim_{\partial x \to 0} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ (3.5)

If the results. 2 and 3 are to have the same value for $f'(z)$ irrespective of the path chosen for δz to tend to zero, then

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y} \quad (3.6a$$

Equating real and imaginary to their corresponding parts.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad (3.6b)$$

These are the Cauchy- Riemann equation

Therefore the necessary condition for $f(z) = wu + iu$ to be regular at $z = z_0$ is that u, v and their partial derivatives are continuous in the neighborhood of $z = z_0$. As in equation (3.6b)

Where a function fails to be regular, a singular point, or singularity occurs, ie where $w = f(z)$ is not continuous where the $Cauchy -$ Riemann test fail.

3.2.1Complex Integration

Differentiation with respect to z in the case of a complex function, since z is a function of two independent variable x *and* $y$ie $z = x + iy$.

In the same way

$$z = x + iy \; and \; w = f(z) = u + iv \quad (3.7a)$$

where$u \; and \; v$ are also function of $x \; and \; y$

also

$$dz = dx + idy = dw = du + idv$$

$$\therefore \int wdz = \int f(z)dz = \int (u + iv)\,(dx + idv)$$

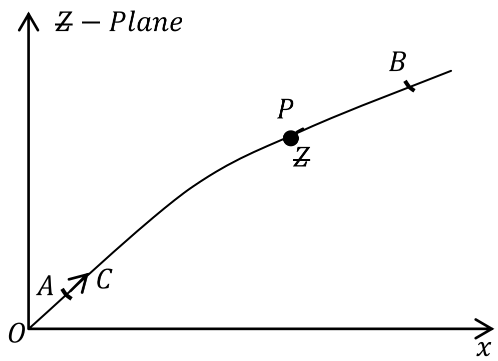
$$= \int [(udx - vdy) + i(vdx + udy)]$$

$$\therefore \int f(z)dz = \int (udx - vdy) + i \int (vdx + udy)$$

Ie we have two real-variable integrals

$\int (udx - vdy) \; and \int (vdx - udy)$ (3.7b)

Because of this introduce contour integral contour integration-line integral in the 2-plane.



If we are summing up $f(z)$ for all such points between $A \; and \; B$, it means that we are evaluating a line integral in the $z - Plane$ between $A(z = z_1)$ and $B(z = z_2)$ along the curve C

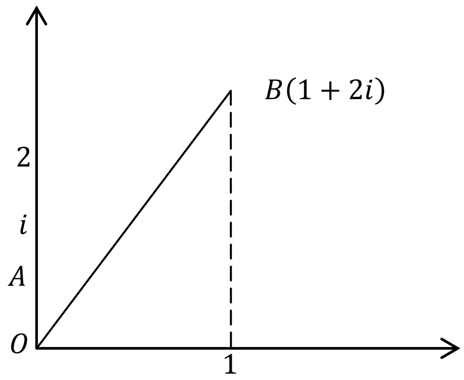
ie$\int_c f(z)dz$ where c is the particular path foining $A \; anad \; B$

the evaluation of line integrals in the complex plane is known a**s**

3.2.2Contour integral

Evaluate integral $\int_c f(z)dz$ where $f(z) = (z - i)^2$ and c is the straight-line joining $A(z = 0)$ to $B(z = 1 + 2i)$

Solution



$$z = x + iy; dz = dx + idy$$

$$f(z) = (z - i)^2$$

$$= [x + i(y-1)]^2$$

$$= x^2 - (y-1)^2 + 2xi(y-1)$$

$$\therefore I = \int [(x^2 - y^2 2y - 1) + i(2xy - 2x)][dx + idy]$$

$$+i\int [(2xy - 2x)dx + (x^2 - y^2 + 2y - 1)dy]$$

Equation of AB is $y = 2x$

$$\therefore dy = 2dx$$

$$\therefore I = \int_o^1 [(x^2 - 4x^2 + 4x - 1)dx - (4x^2 - -2x)2dx]$$

$$+i\int_o^1 [(4x^2 - 2x)dx + (2x^2 - 8x^2 + 8x - 2)dx$$

$$= \int_o^1 (-11x^2 + 8x - 1)dx + i\int_o^1 (-2x^2 + 6x - 2)dx$$

$$= \left[\frac{-11}{3}x^3 + 4x - x\right]_0^1 + i\left[-\frac{2}{3}x^2 + 3x - 2x\right]_0^1$$

$$= -\left(\frac{-11}{3} + 4 - 1\right) + i\left(\frac{-2}{3} + 3 - 2\right)$$

$$= -\frac{-2}{3} + \frac{11}{3} \Rightarrow \frac{1}{3}(-2 + i)$$

Example verify Cauchy's theorem by evaluating the integral

$$\oint_c f(z)\,dz \; where \;\; f(z) = z^z$$

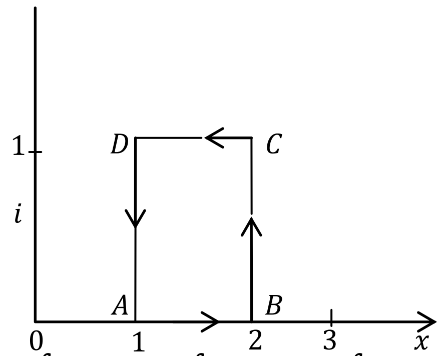
Around the square formed by joining the point $z = 1, z = z, z = 2 + i, z = 1 + i$

Solution

$$z = x + iy$$

$$z^z = x^2 - y^2 + 2xyi$$

$$dz = dx + idy.$$



$$\oint_c f(z)\,dz \oint_c z^z\,dz = \oint_c \{x^2 - y^2 + 2ixy\}\{dx + idy\}$$

$$\Rightarrow \oint_c [(x^2 - y^2)dx - 2xydy] + i\oint_c [2xydx + (x^2 - y^2)dy]$$

(a) $AB; y = 0, dy = 0$

$$\int_{AB} f(z)\,dz = \int_1^2 x^2 dx = \frac{x^3}{3}\bigg|_1^2 \frac{8}{3} - \frac{1}{2} = \frac{7}{3}$$

(b) $BC; x = 2, \therefore dx = 0$

$$\int_{AB} f(z)\,dz = \int_0^1 (4 - y)dy$$

$$= [-2y^2]_0^1 + i\left[4y - \frac{y^3}{3}\right]_0^1 = -2 + i\frac{11}{3}$$

(c) $CD; y = 1, \therefore dy = 0$

$$\therefore \int_{CD} f(z)\, dz = \int_2^1 (x^2 - 1)dx + i\int_2^1 2xdx$$

$$\left[\frac{x^3}{3} - x\right]_2^1 + i[x^2]_2^1 = \frac{-4}{3} - 3i$$

(d) $DAx = 1,, \therefore dx = 0$

$$\int_{DA} f(z)\, dz = \int_1^0 (-2ydy + i\int_1^0 (1 - y^2)\, dy$$

$$= [-y^2]_0^1 + i\left[y + \frac{y^3}{3}\right]_1^0 = 1 - \frac{2}{3}i$$

$$\therefore \oint_c f(z)dz = \frac{7}{3} + \left(-2 + \frac{11}{3}i\right) + \left(\frac{-4}{3} - 3i\right) + \left(1 - \frac{2}{3}i\right) = 0$$

This gives the statement of Cauchy's theorem that if $f(z)$ is regular at every point within and on a simply connected closed curve C, the $\oint_c f(z)dz = 0$

3.2.3 COMPLEXVARIABLES AND OF CONTOUR INTEGRATION

1 Analytic function

Consider $f(z)$ where $z\, is$ complex. The delusive of is

$$f'(z) = \lim_{\Delta z \to \infty} \frac{f(z+\Delta z) - f(z)}{\Delta z} \qquad (3.8)$$

Let $z = x + iy$, $f(z)x + iy) = u(u, y) + i\, v(x, y)$.

If $f'(z)$ exists, then with $\Delta z = \Delta x$, we get $f'(z) = ux + ivx$

And with $\Delta z = i\,\Delta y$, $f'(z = -i\,(uy + ivy) = -ivy + vy$

(1)x(2) Imply that

$$\begin{cases} u_x = v_y \\ v_x = -u_y \end{cases} (3.9)$$

These equations are called the Cauchy-Riemann condition.

Conversely, if u_x, v_x, u_y and v_y exist, satisfy the Cauchy conditions, and are continuous in a neighborhood of z, then $f(z) = +\big(v(x, y)\big)$is differentiable at $i.e. f'(z)$ exists.

To see this, write

$$u(x+a+,y+ay)-u(x,y=\Delta u=u+\Delta x+uy\,\Delta y+f(z)|\Delta z|$$

Where $\lim_{|\Delta z|\to 0}\varepsilon,(z)=0 \quad \Delta z=\Delta x+i\Delta y$

$$v(x+\Delta x,yr\,\Delta y)-v(x,y)=\Delta v=vx\,\Delta x+vy\,\Delta y+\varepsilon_2(z)|\Delta z|$$

Where $\lim_{|\Delta z|\to 0}\varepsilon_2(z)=0$

The above equations are valid in a neighborhood of z since u and v are differentiable at (x,y).

Then, with $\Delta y=\Delta u+i\ \Delta v$, we get

$$\frac{\Delta f}{\Delta z}=(ux+i\,vx)\frac{\Delta x}{\Delta z}+(uy+ivy)\frac{\Delta y}{\Delta z}+\varepsilon_1(z)+\varepsilon_2+(z)$$

$=(u_x+iv_x)\frac{\Delta x}{\Delta z}(-v_x+iu_x)\frac{\Delta y}{\Delta z}+\varepsilon_1(z)+\varepsilon_2+(z)$(3.10a)

In other words, $f'(z)$ exists and is such that $f'(z)=u_x+iv_x=-iy_y+v_y$

Theorem: The function $f(z)=u(x,y)+i\,v(x,y)$ is different a point $z=x+iy$ if and only if the partial derive u_x,v_y,v_x,v_y are continuous and satisfy the Cauchy-Reconditions in a neighborhood of z.

Definition: $f(z)$ is analytic at a point z_0 if $f(z)$ is differentiable inthe neighborhoodof z_0.

$f(z)$is analytic in a region R if it is analy at every point in R.

Remarks: 1/if $f(z)$ is differentiable, the level curve of $u(x,y)=Re\,(f(z))$ and $v(x,y)=$ in $(f(z))$ are orthogonal every point where $f(z)\neq 0$.

Indeed,

A $|f'(z)|^2={u_x}^2+{v_x}^2={u_x}^2+{v_y}^2=|\Delta u|^2$ (3.10b)

$$=v_y^2+{u_x}^2=|\Delta v|^2$$

$i.e.\,\Delta u\neq 0$and$\Delta v\neq 0$ if $f'(z)\neq 0$

$\because \Delta u.\Delta v=u_x\,v_x+u_y\,v_y=v_y(-u_y)+u_y\,v_y=0$

$$i.e.\,\Delta u\perp\ \Delta v.$$

2/ We will see lealer that if f is analytic; the it has derivatives of all orders in the region of analyticity, and that $u = Re(f)$ and $v = Im(f)$ have continuous derivatives of all orders.

So if f is analytic, then uxy and uyx are contain and are therefore equal. Similarly, $u_{xy} = v_{yx}$.

Thus, $u_{xx} + u_{yy} = u_{yx} - v_{xy} = 0$

$$u_{xx} + u_{yy} = -v_{yx} + v_{xy} = 0 \quad (3.11)$$

i.e. the real and imaginary parts of f harmonic functions in the region of limit.

Examples: $f(z) = e^2 = e^{x+iy} = e^x \cos(y) + i\, e^x \sin(y)$

$$u(x,y) = e^x \cos(y) = v_y, \qquad v_y = -\, e^x \sin(y) = -v_x$$

$i.e. f$ is analytic.

$$f(z) = \bar{z} = x = iy \;\; u(x, y = x) \;\; v(x,y) = -y$$

$$u_x = 0 \; v_y = -1 \neq \; u_x \quad (3.12)$$

So $f(z) = \bar{z}$ is nowhere analytic.

Definitions:

$f(z)$ is entire if it is analytic at every point in the complex plane.

A singular point is a point where f fails to be analytic.

Properties:

(i) Products of analytic functions and analytic quotients of analytic functions are analytic unless the denominator vanishes.

(ii) The product rule, the quotient rule and the chain rule apply to

analytic functions.

3.2.3 Complex integration:

We will be interested in integrals of complex valued f on a curve C in the complex plane.

Definitions:

(i) A curve of arc C is simple if it does not intersect itself.

(ii) A contour is an arc consisting of a finite number of connected smoothed arcs.

(iii) A Jordan contour is a simple closed contour.

Considers a parameterization given as: $z = z(t) = x(t) + iy(t) a \leq t \leq b$. Then $\int_C f(z)dz = \int_a^b f\big(z(t)\big)z'(t)dt$ if f is piecewise continuous.

Remarks:

1/ suppose that $f(z)$ is analytic and$F'(z) = f(z)$. Then,

$$\int_C f(z)dz = \int_C F'(dz) = \int_a^b F'\big(z(t)\big)z'(t)dt = [F(z(t))]\,{}_a^b$$

$= F(zb) - F\big(z(a)\big) = F(z_2) - F(z_1)$ (3.13)

Where C is a contour lying in the domain of analyst of with endpoints $z_1$andz_2.

In other words, the integral of f is path-independent.

2/ Note the analogy with vector calculus in the plane.

3/ The fine integral does not depend or the choice of parameterization for the contour C.

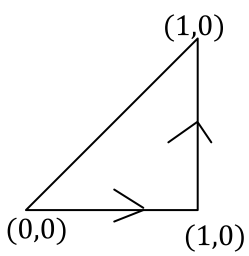
Examples:

1/ Evaluate $\int_C \bar{z}dz$where C_1 is a straight line between (0,0) and (1,1)

$$z = t(1+i) \quad 0 \leq t \leq 1$$

$$\int_C \bar{z}z = \int_C t(1-i)(1+i)dt = \int_a^b 2tdt = [t^2]\,{}_0^1 = 1$$

Evaluate $\int_C \bar{z}dz$ where C_2 is the straight fine from (0,0) to (1,0) and straight line from (1,0) to (1, 1)



$$\int_C \bar{z}\ dz = \int_C^1 t\ dt = \int_0^1 \overline{(1+it)}\, i\, dt = \left[\frac{t^2}{2}\right]{}_0^1 + \int_0^1 (i+t)dt = \frac{1}{2} + i + \frac{1}{2} = 1 + i$$

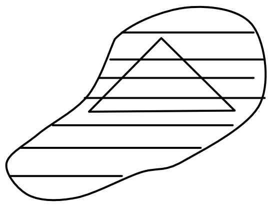
So the integral of $\bar{z}$ is path-dependent. This is due to the fact that $\bar{z}$ is not analytic.

2/ Evaluate $\int_C dz$ along the 2 contours described above since

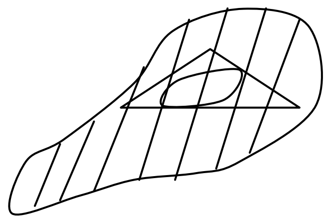
$z = \frac{d}{dz}\left(\frac{1}{2}z^2\right)$, the integral is path-independent and $\int_C dz = zdz = \frac{1}{2}(1+i)^2$.

3.2.5Cauchy's theorem:

Definition: A domain D is simply connected if every is closed contour within it enclose only points of D.



Simply connected



not simply connected

Cauchy's theorem: If a function f is analytic in a *simple* connected domain D, then

$$\oint_C f(z)dz = 0 \quad (3.14)$$

Along any simple closed contour C in D

Proof: 1/ we assume that $f'(z)$ is continuous in D(which is time)$f(z) = u(z) + iv(z); dz = dx + idy$

$\oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C dvx + udy$ (3.15)

Recall Green's theorem: $\boldsymbol{F} = \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix}$vector field then

$\oint_C F.dr = \oint_C \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\int_D \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})$ dxdy If we assume that the partial derivative of u and v are continuous in C and so we can write that

$$\oint_C f(z)dz = \oint_D \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\oint_D \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$$

$= \oint_D (u_y - u_y)dxdy + i\oint_D (u_x - u_x)dx\,dy = 0$ (3.16)

Since D is simply connected (no hole) and the derivative of and v are continuous

2/ In fact there is a more general proof due to Goes which shows that if we only assume f analytic.ie,f'(z) exits in a neighborhood of z, the above theorem is still valid.

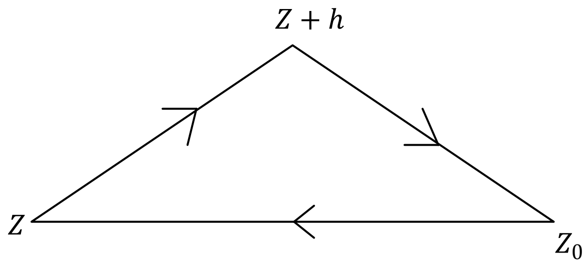
Theorem: If $f(z)$ is continuous in a simply connected domain and if $\int f(z)dz = 0$ for every simple closed curve C lying then there exists a function $F(z)$, analytic in D, such that $F^{'}(z) = f(z)$.

Proof: Consider $Z_0 z\, CD$ and define $F(z) = \int_{Z_0}^{z} f(s)ds$

Then fro h such that $z + h\ \epsilon\, D$,

$$F(z+h) - F(h) = \int_{Z_0}^{z+h} f(s)ds - \int_{z_0}^{z} f(s)ds \quad (3.17)$$

The triangle with vertices (z_0, z, z) makes a closed curve in D and with Cauchy's theorem.



$i.e.$

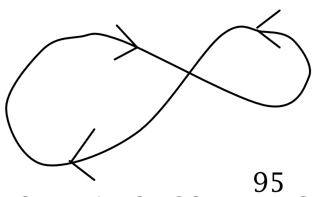
$$-\int_{z_0}^{z} f(z)dz + \int_{z}^{z+h} f(z)dz + \int_{z+h}^{z_0} f(z)$$

$$\int_{z_0}^{z} f(z)dz + \int_{z}^{z+h} f(z)dz + \int_{z+h}^{z_0} f(z)dz$$

Thus, $F(z+h) - F(z) = \int_{z}^{z+h} f(z)dz$

And since f is continuous, $\lim_{n\to\infty} \frac{F(z+h)-F(z)}{h} = f(z)\, F$ is differentiable and $F^{'}(z) = f(z)$ for every z inside D.

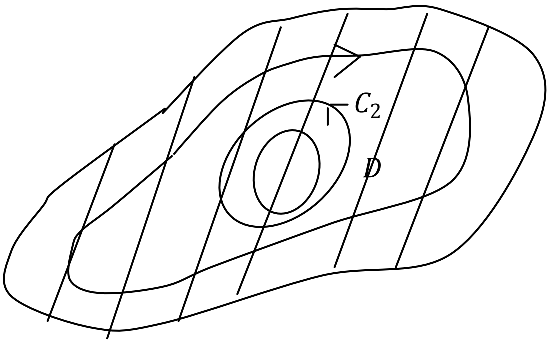
Remarks: 1/what if we have a non-simple contour?

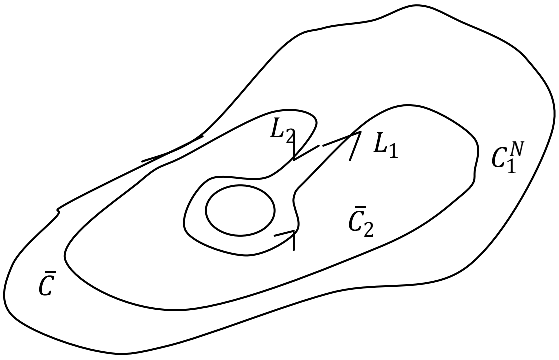


Break the contour into a collection simple contuse (which is possible an as the contour intersects itself a finite of time), and apply Cauchy's the to each simple contour.

2/ Ho to deal with multiply connected demarches?

We introduce cross-cuts which connect his simile closed curves the domain of analycity of f.





Since $\bar{C}$ is ∝ simple closed curve, and since f is analytic in D

$$0 = \int_{\tilde{C}} f(z)dz + \int_{\bar{C}_1} f(z)dz + \int_{L_2} f(z)dz + \int_{\tilde{C}_2} f(z)dz + \int_{L_1} f(z)dz$$

In the limit $L_1 \to L_2$, $\hat{C}_1 \to C_1$ and $\hat{C}_2 \to C_2$, and

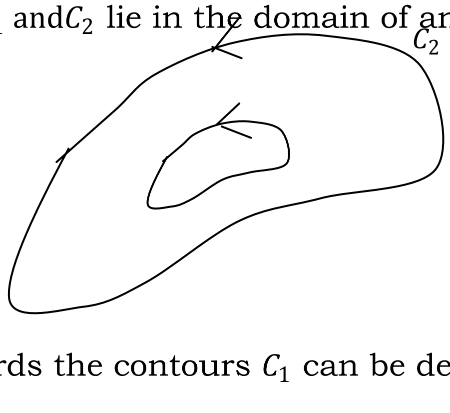
$$\int_{C_1} f(z)dz + \qquad 3.18)$$

$i.e.\ \ 0 = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$ $i.e.$we still have Cauchy's theorem but for the sum of the 2 contours C_1&C. Often, the contours C_1 andC_2 will

have the same orientation and

$$\int_{C_1} f(z)dz = \int_{C_2} f(z) \to 0 \quad (3.19a)$$

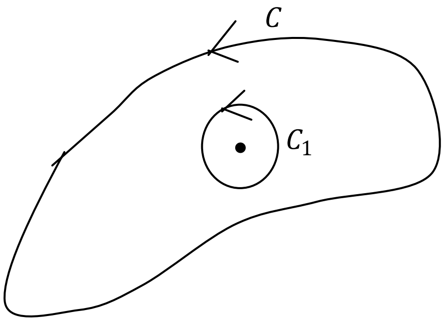
As long as C_1 andC_2 lie in the domain of analycity of D



In other words the contours C_1 can be deformed into the contourC_2.

Example: Evaluate $I = \frac{1}{2\pi i}\oint_C \frac{dz}{(z-\alpha)^m} m = 1{,}2$ where $C$is a simple closed contour, contour counterclockwise if it does not lie in C, $I = 0$

2/ If it lies in C, since f is analytic everywhere but at $z = c$



then$\oint_C\ f(z)dz \oint_{C_1} f(z)dz$

Where C_1 is cancelled, the radius of f is a *thus making* $z = a$ and strictly inside C.

With $z = a = r^{ei\theta}$, we get

$$\oint_{C_1} f(z)dz = \int_0^{2\pi} \frac{1}{r^{eiom}} re^{io} id\theta = \frac{i}{r\mu - 1}\int_0^{2\pi} e^{i(l-m)\theta} d\theta =$$

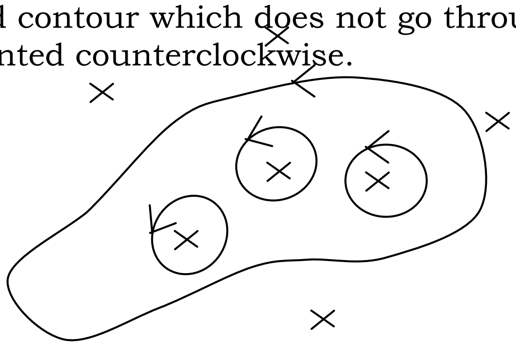
$$\frac{i}{r\mu - 1}\Bigg| \begin{matrix} 2\pi & if & m = 1 \\ 0 & if & m \neq 1 \end{matrix}$$

$i.e. I = \frac{1}{2ri}\int_0^{2\pi} \frac{dz}{(z-\alpha)^m}\Bigg| \begin{matrix} 2\pi & if & m = 1 \\ 0 & if & m \neq 1 \end{matrix}$ (3.19b)

Example: Let $f(z) = A(z - a_1)(z - a_2)(z - a_n)$ be a polynomial of degree n

Then, $\frac{P'(z)}{p(z)} = \frac{1}{z-\propto_1} + \frac{1}{z-\propto_2} + \cdots + \frac{1}{z-\propto_n}$

and therefore $\frac{1}{2\pi i}\int_C \frac{P'(z)}{p(z)} dz =$ number of roots of P lying where C is a simple closed contour which does not go through any of the roots, and which is oriented counterclockwise.



3.2.6 Cauchy's integral formula and its derivatives

Theorem (Cauchy's integral formula): Let $f(z)$ be analytic intern to and on a simple closed contour C. then, at any into point ,

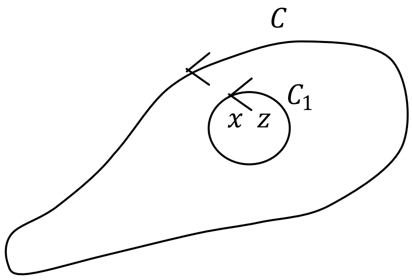
$$f(z) = \frac{1}{z-\propto_1} \oint_C \frac{P'(z)}{p(z)} dz. \qquad (3.20a)$$

Proof:

Take z inside C and draw a ci of radius r centered at z shanty.

Then

$$\int_C \frac{f(y)}{y-z} dy = \int_{C_1} \frac{f(y)}{y-z} dy \qquad (3.20b)$$



$$\oint_C \frac{f(\xi)}{\xi - z} d\xi = \oint_{C_1} \frac{f(z)}{\xi - z} + \oint_{C_1} \frac{f(\xi) - f(z)}{\xi - z} d\xi$$

$$= f(z)2\pi i + \oint_{C_1} \frac{f(\xi) - f(z)}{\xi - z} d\xi$$

$$\left| \oint_{C_1} \frac{f(\xi) - f(z)}{\xi - z} d\xi \right| \leq \oint_{C_1} \frac{|f(\xi) - f(z)|}{|\xi - z|} dy$$

If r is chosen small enough, $|f(\xi) - f(z)| < \varepsilon$ and

$$\oint_{C_1} \frac{|f(\xi) - f(z)|}{|\xi - z|} |d\,\xi| < \frac{\varepsilon}{r} \oint_{C_1} |d\xi \;| = \frac{\varepsilon \pi r \varepsilon}{r} = 2\pi\varepsilon$$

In other words, $\forall\, z > 0 \quad \ni r_o \quad r < \ni r_o \; \rightarrow \left| \oint_{C_{1(r)}} \frac{f(\xi) - f(z)}{\xi - z} d\xi \;\right|$

So $\lim_{n\to 0} \oint_{C_{1(r)}} \frac{|f(\xi\ ) - f(z)|}{\xi\ - z} dz = 0$ (3.21)

Theorem: If $f(z)$ is analytic interior to and on a simples closed contour then all the derivatives $f^{(h)}(z), h = 1,2,\cdots$ exist in the domain intoed to C and.

$$f^{(h)}(z) = \frac{h\ !}{2\pi\ i} \oint_C \frac{f(\xi\ )}{(\xi\ - z)^{h+1}} d\xi$$

Proof: $1/h = 1$ consider $\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi\ i} \frac{1}{h} \oint_c \frac{f(z)}{\xi\ - (z+h)} - \frac{1}{\xi}$

$$i.e. \frac{f(z+h) - f(z)}{h}$$

$$= \frac{1}{2\pi hi} \oint_C \frac{f(z)h}{[\xi\ - (z+h)][\xi\ - z]} d\xi\ = \frac{1}{2\pi\ i} \frac{f(z)h}{[\xi\ - (z+h)](\xi\ - z)}$$

$$= \frac{1}{2\pi hi} \oint \frac{f(\xi\ )}{(\xi\ - z)^2 - h(\xi - z)} d\xi$$

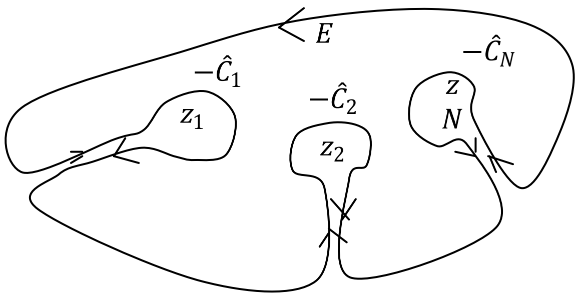
3.2.7 Cauchy's residue theorem and contour integration

Theorem: Let C be a simple closed contour inside and on which f is analytic, except for a finite number of isolated singular points $z_1, \cdots, z_n$ in the interior of the region bounded by C. Then,

$$\oint f(z) dz = 2\pi i \textstyle\sum_{j=1}^{N} a_j \qquad (3.22)$$

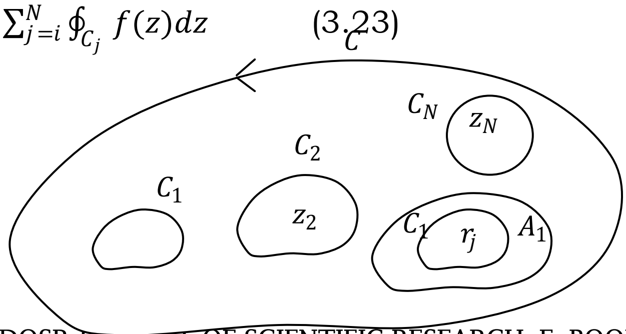
Where a_j is the residue of $f(z)$at$z = z_j$

Proof:



By introduces crosscuts as show on the figure, we have

$$\oint f(z) dz = 2\pi i \textstyle\sum_{j=i}^{N} \oint_{C_j} f(z) dz \qquad (3.23)$$



But since f is analytic in an anulus centered at each z_i, it has a laurent expassion in A_i given by

$$f(z) = \sum_{n=-\infty}^{+\infty} C_n^i (z - z_i)^n$$

Where $C_n^i = \frac{1}{2\pi i} \oint_{C_i} f(z)dz = a_i$ is the residue of f a z_i

$$\left[\text{Recall: } C_n^i \frac{1}{2\pi i} \oint_{C_i} \frac{f(z)dz}{(z - z_i)^{n+1}}\right]$$

Therefore, $$\oint_C f(z)dz = \sum_{j=i}^{N} a_j \qquad (3.24)$$

Examples:

1/ Evaluate$i_h = \frac{1}{2\pi i} \oint_C z^h \, dh \quad h \in Z$, where C is the unit circle centered at the origin.

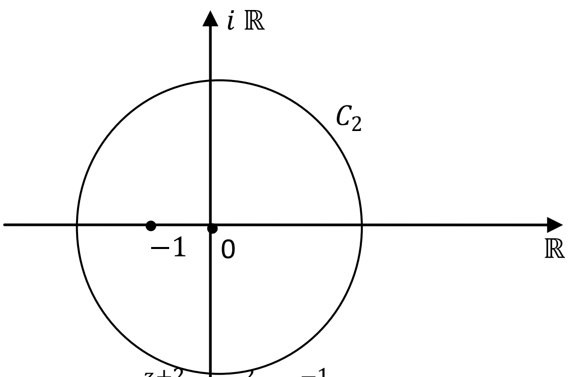
If $h \geq 0$or $h \leq -2$, the residue of z^h is zero, so $i_h = 0$

If $h = -1$ $z^h = \frac{1}{z}$ and the residue is 1, so that $i_h = 0$

Therefore,

$$\boxed{\frac{1}{2\pi i} \oint_C z^h \, dz = \delta_{h1-i} = \begin{vmatrix} 0 \text{ } if \text{ } h \neq -1 \\ 1 \text{ } if \text{ } h = -1 \end{vmatrix}}$$

2/ Evaluate $I = \oint_{C_2} \frac{2+2}{z(z+1)} dz$ where C_2 is the circle of radius 2 centred at the origin.



We write $\frac{z+2}{z(z+1)} = \frac{2}{z} + \frac{-1}{z+1}$

and$I = \oint_{C_2} \frac{2+2}{z(z+1)} dz = 2\pi i(2-1)$

$i.e.$ $\boxed{I = 2\pi i}$

3/ suppose $f(z) = \frac{\emptyset(z)}{(z-z_0)^m}$ where $\emptyset(z)$ is analytic in a neighborhood of $z = z_0$ and m is a positive integer.

What is the reside of $f$at z_0?

Since $\emptyset$ is analytic, it has a Taylor series near $z = z_0$,

$i.e.\emptyset(z) = \sum_{h=0}^{\infty} \frac{1}{h\,!} \emptyset^{(h)}(\mathrm{z_0})(\mathrm{z} - \mathrm{z_0})^{\mathrm{h-m}}$

The residue of f at z_0 is then $\boxed{\frac{\emptyset^{(m-1)}(z_0)}{(m-1)!} = C_{-1}}$ for a simple pole,

$$C_{-1} = \emptyset(z_0) = \lim_{z \to z_0} [f(z)(z - z_0)]$$

Similarly, if $f(z) = \frac{N(z)}{D(z)}$ and $D(z)$ has a simple pole at $z = z_0$, the residue of f at $z = z_0$ is $\frac{N(z)}{D(z)}$, where

$(\not{z})$and$d(z)$ are both analytic in a neighborhood of z_0

- Evaluate $I = \frac{1}{2\pi i} \oint_C \frac{3z+1}{z(z-1)^3} dz$ where C is the circle of radius z at the origin.

$$I = -1 + \frac{1}{2\,!} \frac{d^2}{dz^2} \left(\frac{3\,z + 1}{z} \right) |_{z=1} = -1 + \frac{1}{2} \frac{d^2}{dz^2} |_{z=-1}$$

$= -1 + \frac{1}{2}(2) = 0$ $\boxed{0 = I}$

- Evaluate $I = \frac{1}{2\pi i} \oint_C \, cos\,z\,dz$ where C is the unit circle centered at the origin. Not $z = \frac{\cos z}{\sin z}$

$Sin\,z$ vanisheo at $z = n\pi$ so $cosz = \frac{\cos z}{\sin z}$ has a singutanitly at $z = 0$ in the unit disk.

The residue is given by $\frac{\cos \ (0)}{\sin'(z)|_{z=0}}$

Thus, $\frac{1}{2\pi i} \oint_C \, cot\,z\,dz = 1$

Evaluation of indefinite integrals on the real line,

In this case, we are interested in integrals of the form

$$I = \int_{-\infty}^{+\infty} f(x)dx$$

(I) converges if $\lim_{c\to+\infty} \int_{-l}^{a} f(x)dx$ and $\lim_{c\to+\infty} \int_{-l}^{a} f(x)dx$ are both convergent.

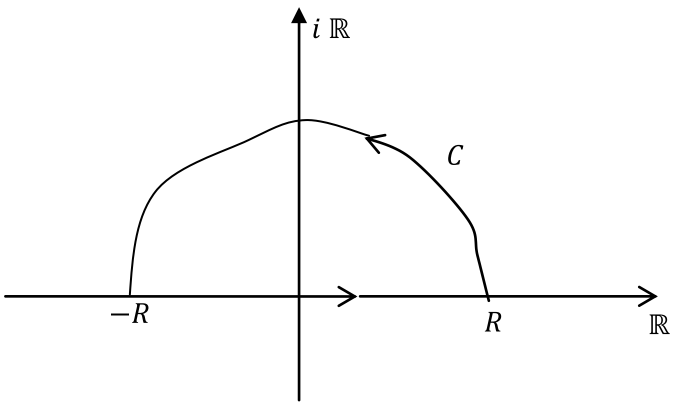
We will often consider integrals of the form

$$I_p = \lim_{c\to+\infty} \int_{-l}^{l} f(x)dx$$

Which is referred to as the Cauchy principal value at infinity of f.

Note that I_p may converge aqnd I diverge.

The idea is to introduce a contour in the upper of lower half planes as shown below:



And compute $\oint_C f(z)dz$ using Cauchy's residue theorem.

Often, $\int_{C_R} f(z)dz \to 0$ are $r \to \infty$, where C_R is the are $z = R^{ei\theta}$ $0 \leq \theta \leq \pi$. As a consequence,

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{R\to\infty} \oint_C f(z)dz$$

It, of course, $\int_{-\infty}^{+\infty} f(x)dx$ converge. We often use the following lemma, called Jardan's lemma.

Jordan's lemma: Suppose that $f(z) \to 0$ untformly as $R \to \infty$ on the are $C_R^{:} z = R^{ei\theta}\ 0 \le \theta \le \pi$ (i.e. $\left|f(R^{ei\theta})\right| \le G(R)$ and $G(R) \to 0$ ad $R \to \infty$). Then,

$$\lim_{R \to \infty} \int_{C_R} e^{ihz} f(z)dz = 0 \qquad h > 0$$

Proof: $\left|\int_{C_R} e^{ihz} f(z)dz\right| \le \int_0^\pi \left|e^{ih(\cos\theta + i\sin\theta)R} f\left(R^{ei\,\theta}\right) R^{ei\theta}\, id\theta\right|$

$$= \int_0^\pi e^{-h\,R\,\sin\,\theta} \left|f(R^{ei\theta})\right| R\, d\theta$$

$\le \int_0^\pi e^{-h\,R\,\sin\,\theta} G(R)R\, d\theta$ (3.25)

Since $sin\theta \ge \frac{2\theta}{\pi}$ for $0 \le \theta \le \frac{\pi}{2}$, we have

$$\left|\int_{C_R} e^{ihz} f(z)dz\right| \le \int_0^{\pi/2} e^{-h\,R\frac{2\,\theta}{\pi}} G(R)R\, d\theta + \int_{\pi/2}^{0} e^{+h\,R\sin(\theta-\pi)}\, G(R)R\, d\theta$$

$$\int_{\pi/2}^{\pi} e^{h\,R\,\sin\,(\theta-\pi)} G(R)R\, d\theta = \int_{-\pi/2}^{0} e^{h\,R\,\sin\,u}\, G(R)R\, d\theta = \int_0^{\pi/2} e^{-h\,R\,\sin\,u} G(R)\; Rd\theta$$

Thus $\left|\int_{C_R} e^{ihz} f(z)dz\right| \le 2\int_0^{\pi/2} e^{-h\,R\frac{2\theta}{\pi}}\, G\,(R)R\, d\theta$

$= \left[-\frac{\pi}{h}\, G(R)e^{-h\,R\frac{2\theta}{\pi}}\right]_0^{\pi/2}$ (3.26)

$= \frac{G(R)\pi}{h}(1 - e^{-h\,R}) \to 0$ as $\to \infty$

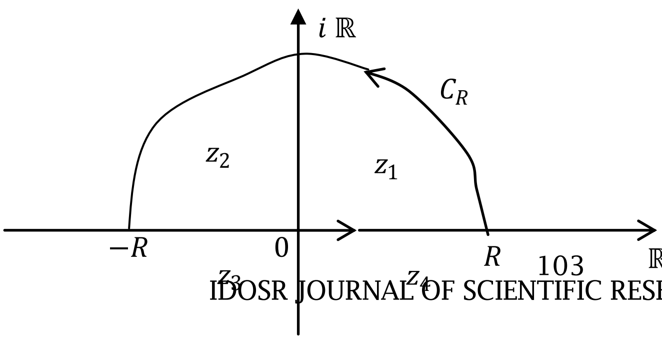
Since $G(R) \to 0$ as $R \to \infty$.

Examples:

1/Evaluate $I = \int_{-\infty}^{+\infty} \frac{x^2}{x^u+1} dx$

(Note that I converges. Jordan's lemma does not apply)

$$x^4 + 1 = 0(x)z^4 - 1(x\,)z^2 = \pm i = e^{\pm i^{\pi}/2}(=)z = e^{\left(\frac{\pi}{1}+n\frac{\pi}{2}\right)}$$



$$\left|\int_{C_R} \frac{z^2\, d^z}{z^2+1}\right| \leq \frac{\pi\, R^3}{R^4-1} \to as\; R \to +\infty$$

(This is not Jordan's lemma). Then,

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^u+1}dx = \int_C \frac{z^2}{z^4+1}dz.$$

We now use Cauchy's residue of $f(z) = \frac{z^2}{z^4+1}$at $z_1 = \frac{\sqrt{2}}{2}(1+i)$ and $z_2 = \frac{\sqrt{2}}{2}(1 \pm 1)$, respeclirely.

$$a_1 = \frac{e^{2i\,\pi/4}}{4e^{3i\,\pi/4}} = \frac{1}{4}e^{-i\pi/4} \qquad a_2 = \frac{e^{2i\,3\pi/4}}{4e^{3i\,3\pi/4}} = \frac{1}{4}e^{-3ci\pi/4}$$

$$a_1 + a_2 = \frac{1}{4}\left(e^{-i\pi/4} + e^{-3i\pi/4}\right) = \frac{1}{4}(1-i-1-i)\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4}i$$

Thus,

$$\boxed{\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}dx = \frac{\sqrt{2}}{2}\pi = \frac{\pi}{\sqrt{2}}}$$

2/Evaluate I=R $\int_{-\infty}^{+\infty} \frac{x\sin(\propto x)\cos(\beta x)}{x^2+\gamma^2}dx \quad \gamma > 0$

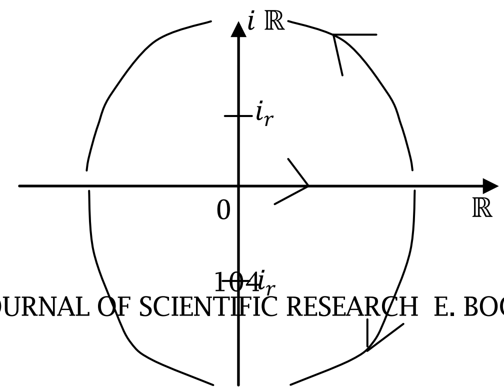
$\propto, \beta$ real,and $/\propto/\neq/\beta/$

$\sin(\propto x)\cos(\beta x) = \frac{1}{2}(\sin[(\propto -p)x] + \sin[(\propto | + p)x])$. We can then write x as

$I = \int_{-\infty}^{+\infty} x \frac{(\sin[(\propto -p)x] + \sin[(\propto | + p)x])}{x^2+r^2}dx = Im(J)$ Where

$$J - J_1 + J_2 = \int_{-\infty}^{+\infty} x\frac{e^{i(x-\beta)x}}{x^2+r^2}dx + \int_{-\infty}^{+\infty} \frac{e^{i(x+\beta)x}}{x^2+r^2}$$

We now use complex integration to evaluate $J_1$and J_2.



If we use a semi-circular contour, we will have to apply Jordan's lemma. If $\propto -\beta > 0$ we use the contour C_R^+ in the upper half plane and get.

$$\int_{C^+} \frac{ze^{i(\propto-\beta)z}}{z^2+\gamma^2} dz = 2\pi i = Res\left(\frac{ze^{i(\propto-\beta)z}}{(z+i\gamma)(z-i\gamma)}, i\gamma\right)$$

$$= 2\pi \frac{i\gamma e^{-(\propto-\beta)z}}{2i\gamma} i\pi\, e^{-(\propto-\beta)z}$$

But if $\propto -\beta < 0$, we have to use the contour C_R^- in the lower half–plane.

Then,

$$\int_{C^-} \frac{ze^{i(\propto-\beta)z}}{z^2+\gamma^2} dz = -2\pi i\, Res\left(\frac{ze^{i(\propto-\beta)z}}{(z+i\gamma)(z-i\gamma)} - i\gamma\right)$$

$$= 2\pi \frac{-i\gamma e^{-(\propto-\beta)z}}{-2i\gamma} = -i\pi\, e^{+(\propto-\beta)z}$$

Therefore, $J_1 = sgn\,(\propto -\beta)\pi i e^{-|\propto-\beta|\gamma}$

Similarly, $J_2 = sgn\,(\propto +\beta)\pi i e^{-|\propto-\beta|\gamma}$

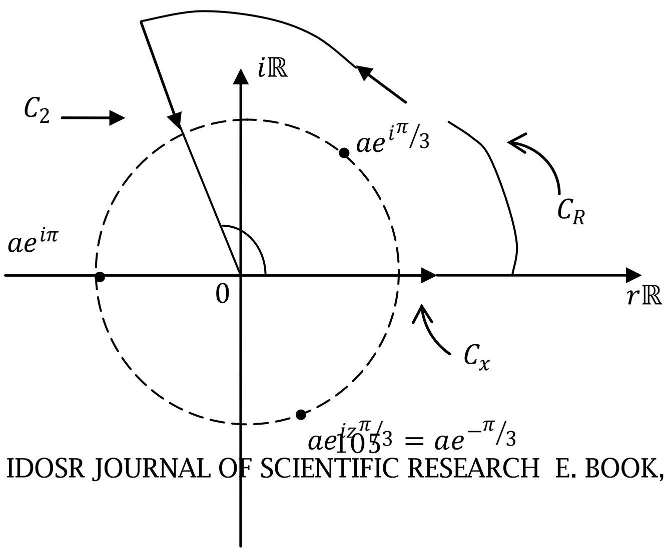
Therefore

$$\boxed{I = \pi\, sgn(\propto +\beta)e^{-|\propto-\beta|\gamma} + \pi\, sng(\propto +\beta)e^{-|\propto-\beta|\gamma}}$$

3/ Evaluate

$$I = \int_0^\infty \frac{dx}{x^3+a^3} \qquad a > 0$$

$$x^3 + a^3 = 0 (=) z^3 = -a^3 = a^3 e^{i\pi} (=) z = a\, e^{i\frac{\pi+2n\pi}{3}}$$



We cannot use a semicircular contour because $\frac{1}{z^3+a^3}$ has a singularly at $z = -a$.

On c_R,

$$\left|\int_{C_R} \frac{dz}{z^3 + a^3}\right| \le \int_{c_R} \frac{|dz|}{R^3 - a^3}$$

$$\le \frac{2\pi R}{R^3 - a^3} \underset{R\to\infty}{\to 0}$$

As $R \to \infty, \int_{C_L} \frac{dz}{z^3+a^3} \to \int_{C_x} \frac{dz}{x^3+a^3}$

How to choose C_L? on $C_L, z = re^{i\theta_L}$

$$\int_{C_x} \frac{dz}{z^3 + a^3} = \int_{+\infty}^{0} \frac{dr\ e^{i\theta_2}}{r^3 e^{3i\theta_2} + a^3}$$

If we pitch $\theta_L = \frac{3\pi}{3}$, $r^3\ e^{3i\theta_L} = r^3\ e^{2\pi i} = r^3$ and

$$\int_{C_L} \frac{dz}{z^3 + a^3} = -\int_{0}^{+\infty} \frac{e^{i2\pi/3}}{a^3 + r^3} dr = -\,e^{i2\pi/3}\ I.$$

Thus, $2\pi i\ Res\left(\frac{1}{z^3+a^3}, ae^{i\pi/3}\right) = I\left(1 - e^{i2\pi/3}\right)$

$$i.e. I\left(1 - e^{i2\pi/3}\right) = 2\pi i \quad \frac{1}{3\left(ae^{i\pi/3}\right)^2} = \frac{2\pi i}{3\ a^2} e^{-2i\pi/3}$$

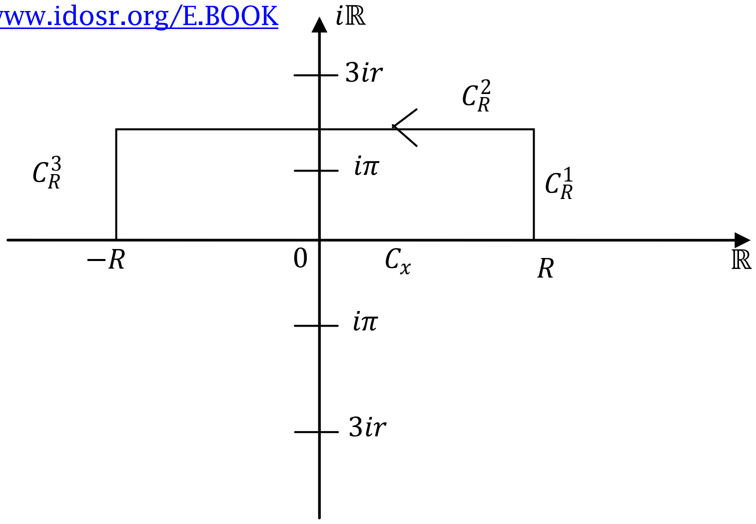
$$i.e. I = \frac{2\pi i\ e^{-2i\pi/3}}{3\ a^2\left(1 - e^{i2\pi/3}\right)} = \frac{2\pi i}{3\ a^2} \frac{e^{-\pi i}}{e^{-\pi i/3} - e^{\pi i/3}} = \frac{+2\,\pi i}{3a^2\ 2i \sin\left(\frac{\pi}{3}\right)}$$

$$= \frac{\pi}{3a^2\frac{\sqrt{3}}{2}} = \boxed{\frac{2\pi}{3\sqrt{3}a^2} = I}$$

4/ Evaluate

$$I = \int_{-\infty}^{+\infty} \frac{eP^x}{1 + e^x} dx \quad 0 < Rep < 1$$

$$1 + e^2 = 0(=)e^2 = -1 = e^{i\pi}\,(=)\ z = i(\pi + 2n\pi)$$



We don't want a contour which goes to∞ in the direction of the imaginary axis since we would have an infinite number of singularities inside the contour.

So we take a rectangular contour $C = C_x U C_R^1 U C_R^2 U C_R^3$.

On $C_R^1, z = R + iy \;\; 0 \leq y \leq 2\pi$

$$\int_{C_R^1} \frac{eP^z}{1+e^x} dz = \int_0^{2\pi} \frac{eP^R e^{-ipy}}{1+e^R e^{-iy}} idy, \text{and}$$

$$\left| \int_{C_R^1} \frac{eP^z}{1+e^z} dz \right| \leq \int_0^{2\pi} \frac{eP^R}{1+e^R} dy = \frac{2\pi \, e^{PR}}{1+e^R} \underset{R \to \infty}{\to 0}$$

Similarly, $z = R + iy$ onC_R^3 and

$$\left| \int_{C_R^3} \frac{eP^z}{1+e^z} dz \right| \leq \frac{e^{-PR}}{1+e^{-R}} 2\pi \underset{R \to 0}{\longrightarrow 0}$$

On $C_R^2, z = +x + 2\pi i$ and

$$\int_{C_R^2} \frac{eP^z}{1+e^x} dz = \int_{+R}^{-R} \frac{eP^R e^{-ipy}}{1+e^{+x} e^{-iy}} dx = -e^{2\pi ip} \int_{-R}^{+R} \frac{eP^x}{1+e^x} dx$$

Therefore,

$$\oint_C \frac{eP^z}{1+e^x} dz = \left(1 - e^{2\pi ip}\right) \int_{-\infty}^{+\infty} \frac{eP^R}{1+e^{+x}} dx$$

And with Cauchy's residue theorem,

$$\oint_C \frac{eP^z}{1+e^x} dz = 2\pi i \; Res\left(\frac{eP^z}{1+e^x}, 2\pi i\right) \frac{e^{ip\pi}}{e^{i\pi}}$$

$$= 2\pi i \;\; e^{i(p-i)\pi}$$

Therefore,

$$\oint_C \frac{eP^z}{1+e^x}dx = \frac{2\pi i\ e^{i(p-i)\pi}}{1-e^{2\pi ip}} = \frac{2\pi i\ e^{-i\pi}}{e^{-i\pi p} - e^{p\pi}}$$

$$= \frac{2\pi i\ e^{-i\pi}}{-2i\sin(p\pi)} = \frac{\pi}{\sin(p\pi)}$$

$$i.e. \oint_C \frac{eP^z}{1+e^x}dx = \frac{\pi}{\sin(p\pi)}$$

5/ Evaluate

$$\int_{-\infty}^{+\infty} \frac{e^{Px} - e^{px}}{1-e^x}dx \quad 0 < p, q < 1$$

Note that though $1 - e^x$ vanishes at $x = 0$, this integral is convergent. We will introduce 2 integrals,

$$J_P = \int_{-\infty}^{+\infty} \frac{e^{Px}}{1-e^x}dx \ \ and\ P_q = \int_{-\infty}^{+\infty} \frac{e^{Px}}{1-e^x}dx$$

where$f$ means the Cauchy principal value. It is defined as:

$$\lim_{z\to 0} + \left(\int_{-}^{+\infty} + \int_{-\infty}^{+\infty} \right) f(x)dx = \int_{-\infty}^{+\infty} f(x)dx$$

In other word,

$\int_{-\infty}^{+\infty} \frac{e^{Px}}{1-e^x}dx$ is not convergent but the divergingContribution cancel out so that the limit as $\delta \to 0 +$ of $\left(\int_{-\infty}^{+\infty} f(x)dx + \int_{\varepsilon}^{+\infty} f(x)dx\right)$ exists.

We will use contour integration to evaluate each of the Cauchy principal value integrals.

$$1 - e^z 0 (=) e^z = 1\ (=)\ z = 2i\pi n,\ \ n\epsilon 2$$

So $\frac{e^{Pz}}{1-e^z}$ singutation at $z = 2i\pi n,\ \ n\epsilon 2$.

3.2.8 POLAR AND RESIDUES

Definition:

(i) If $f(z)$ (or any single-valued branch of $f(z)$ is analytic in the region $0 < |z - z_0| < R$ and not at z_0, then $z = z_0$ is an isolated singular point of $f(x)$.

(ii) If $z = z_0$ is an isolated singular point and if $f(z)$ is bounded, then $z = z_0$ is a removable singularity.

The lament since of f is $f(z) = \sum_{n=\infty}^{+\infty} C_n (z = z_0)^n$ with $0 = C_{-i}, i > 1, f(z)$ is power series. This expansion is valid for every $|z = z_0| < R$ except at $z = z_0$. But we can make of f analytic in $|z = z_0| < R$ by redefining $f(z_0) = C_0$.

(iii) If we can write $f(z) = \frac{d(z)}{(z=z_0)^n}$ in a neighborhood of $z = z_0$ with $\emptyset$ analytic and$\emptyset(z_0) \neq 0$, then $z = z_0$ is a pole at orders nof$f$. If $n = 1$, $z = z_0$ is a simple pole of f.

The coefficient C_{-n} is called the strength of the pole.

(iv) If the principal part of $f(z)$ near $z = z_0$ has an infinite number of terms, then $at\, z = z_0$ hasan essential singularity or an essential singular point.

(v) A function which has only poles in the complex plane is called ageomorphic function.

(vi) The residue of $f(z)$ at $z = z_0$ is the coefficient C_{-1} of the Laurent series expansion of f abut $z = z_0$

Example: 5

Find the laurent expansion of $f(z) = \frac{1}{1+z}$ for $|z| > 1$

$\frac{1}{1+z} = \frac{1}{z\left(\frac{1}{2}+1\right)}$. Since $\frac{1}{\frac{1}{2}+1} = \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n$, we get

$$\frac{1}{1+z} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{-n} = \sum_{n=0}^{\infty}(-1)^n z^{(n+1)}|z| > 1$$

Note that the Laurent expansion for $|z| > 1$would be $f(z) = \sum_{n=0}^{\infty}(-1)^n z^{-n}$

Example 6 Describe the singularities of the function $f(z) = \frac{z^2-2z+1}{2(Z+1)3} f(z) = \frac{(2-1)^2}{z(z+1)^3}$ has a simple pole at $z = 0$ triple→ $z = -1$

Near $z = 0, f(z) = \frac{1}{z}(1 - 2z + z^2)(1 - 3z + \cdots)$

$$= \frac{1}{2}(1 - z + \cdots) = \frac{1}{2} - 5 + \cdots$$

Near $z = -1, f(z) = \frac{1}{(z+1)^3}\frac{(z+1-2)^2}{(z+1-1)} = \frac{1}{(z+1)^3}\frac{1}{2}((z+1)^2 - 4(z+1) + 4)$

$$i.e f(z) = \frac{-1}{(z+1)^3}(1 + (z+1) + \cdots)(4 - 4(z+1) + \cdots)$$

$$= -\frac{1}{(z+1)^3}(4 - 0(z+1) + \cdots) = -\frac{4}{(z+1)^3} + \frac{0}{(z+1)^2} + \cdots$$

Example7 Describe the behavior of the function $f(z) = \frac{z+1}{z \sin z}$

Near $z = 0$

$$\sin z = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!} = z - \frac{z^3}{6} + \cdots$$

$$f(z) = \frac{z+1}{z\left(z - \frac{z^3}{6} + \cdots\right)} = \frac{1}{z^2}(z+1)\left(1 + \frac{z^3}{6} + \cdots\right) = \frac{1}{z^2} + \frac{1}{z} + \frac{z^3}{6} + \cdots$$

So $f(z)$ has a double pole at $z = 0$ with strength 1.

Example 8Discuss the pole singularities of $f(z) = \frac{z^{1/2} - 1}{z - 1}$

$$f(z) = \frac{(t+1)^{1/2} - 1}{t} \text{where } \ z - 1 = t$$

If we take the positive branch of the square root, we have

$$\sqrt{1+t} = 1 + \frac{1}{2} + -\frac{1}{3}t^2 + \cdots$$

And $f(z) = \frac{\frac{t}{2} - \frac{1}{3}t^2 + \cdots}{t} - \frac{1}{2} - \frac{1}{8}t + \cdots = \frac{1}{2} - \frac{1}{8}(z-1) + \cdots$

If we take the negative branch of $z^{1/2}$, we have

$$z^{1/2} = -\sqrt{1+t} = 1 + \frac{1}{2} + -\frac{1}{3}t^2 + \cdots$$

And $f(z) = \frac{\frac{t}{2} - \frac{1}{3}t^2 + \cdots}{t} = -\frac{2}{t} - \frac{1}{2} + \frac{1}{8}t - \cdots$

$$= -\frac{2}{z-1} - \frac{1}{2} + \frac{1}{8}(z-1) - \cdots$$

So depending on the branch we choose, $f(z)$ is other analytic in the neighborhood of z = 1 or has a simple pole at z = 1 with strength 2.

6/ The residue of at $z = z_0$ is the coefficient C_1 of the Laurent series expansion of f about $z = z_0$.

Examples:

9 Find the Laurent expansion of $f(z) = \frac{1}{1+z}$ for

$|z| > | \; \frac{1}{1+z} = \frac{1}{z\left(\frac{1}{2}+1\right)}$ since $\frac{1}{\frac{1}{z}+1} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n$, we get

$$\frac{1}{1+z} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n e^{-n} = \sum_{n=0}^{\infty}(-1)^n z^{-(n+1)} |z| > |.$$

Note that the Laurent expansion for $|z| < |$ world be

$$f(z) = \sum_{n=0}^{\infty}(-1)^n e^{-n}$$

10 Describe the singularities of the function $f(z) = \frac{z^2 - 2z + 1}{2(z+1)^3}$

$f(z) = \frac{(z-1)^2}{2(z+1)^3}$ has a simple pole at $z = 0$ triple $z = -1$

Near $z = 0,\; f(z) = \frac{1}{z}\left(1 - 2z + z^2\right)(1 - 3z + \ldots)$

$$= \frac{1}{z}(1 - 5z + 111) = \frac{1}{2} - 5 + \ldots$$

Near $z = -1,\; f(z) = \frac{1}{(2+1)^3}\frac{(z+1-2)^2}{(z+1-1)} = \frac{1}{(2+1)^3}\frac{1}{z}\left((z+1)^2 - 4\left((z+1)^2 + 4\right)\right)$

i.e. $f(z)=\frac{-1}{(z+1)^3}(1+(z+1)+...)(4-4(z+1)+...)$

$$=-\frac{1}{(z+1)^3}(4-0(z+1)+...)=-\frac{4}{(z+1)^3}+\frac{0}{(z+1)^2}+...$$

11 Describe the behaviour of the function $f(z)=\frac{z+1}{z\sin t}$

Near $z=0.$

Sin $f(z)=\frac{z+1}{z\left(z-\frac{z^3}{6}+1n\right)}=\frac{1}{z^2}(z+1)\left(1+\frac{z^2}{6}-11\right)=\frac{1}{z^2}+\frac{1}{2}+\frac{1}{6}+...$

$$f(z)=\frac{z+1}{z\left(z-\frac{z^3}{6}+1n\right)}=\frac{1}{z^2}(z+1)\left(1+\frac{z^2}{6}-11\right)=\frac{1}{z^2}+\frac{1}{2}+\frac{1}{6}+...$$

So $f(z)$ has a double pole at $z=0$ with strength 1.

12 Discuss the pole singularities of $f(t)=\frac{z^{1/2}-1}{z-1}$

$$f(z)=\frac{(t+1)^{1/2}-1}{t}\text{where } z-1=t$$

If we take the positive branch of the square root, we have

$\sqrt{1+t}=1+\frac{1}{2}+-\frac{1}{8}t^2+...$and $f(z)=\frac{\frac{t}{2}-\frac{1}{8}t^2+...}{t}-\frac{1}{2}-\frac{1}{8}t+...=\frac{1}{2}-\frac{1}{8}(z-1)+...$

If we take the negative branch of $z\frac{1}{2}$, we have

$$z^{1/2}=-\sqrt{1+t}=-1-\frac{1}{2}+\frac{1}{8}t^2-...$$

and $f(z)=\dfrac{-2-\frac{1}{2}+\frac{1}{8}t^2-...}{t}=-\dfrac{2}{t}-\dfrac{1}{8}t-...$

$$=-\frac{2}{z-1}-\frac{1}{2}+\frac{1}{8}(z-1)-...$$

So depending on the branch we choose, $f(z)$ is other analytic in the neighborhood of $z=1$ or has a simple pole at $z=1$ with strength -2.

Examples:

13 Find the Laurent expansion of $f(z)=\dfrac{1}{1+z}$ for

$|z|>|$ $\dfrac{1}{1+z}=\dfrac{1}{z\left(\frac{1}{2}+1\right)}$. Since $\dfrac{1}{\frac{1}{z}+1}=\sum_{n=0}^{\infty}\left(1\frac{1}{z}\right)^n$, we get

$\dfrac{1}{1+z}\dfrac{1}{z}\sum_{n=0}^{\infty}(-1)^n\ z^{-(n+1)}\ \ |z|>|.$

Note that the Laurent expansion for $|z|>|$ would be

$$f(z)=\sum_{n=0}^{\infty}(-1)^n e^{-n}$$

14 Describe the singularities of the function $f(z)=\dfrac{z^2-2z+1}{2(z+1)^3}$ $f(z)=\dfrac{(z-1)^2}{z(z+1)^3}$ has a simple pole at $z=0$ triple $z=-1$.

Near $z=0,\ f(z)=\dfrac{1}{z}\left(1-2z+z^2\right)(1-3z+...)$

$$=\frac{1}{z}(1-5z+111)=\frac{1}{2}-5+...$$

Near $z=-1, f(z)=\frac{1}{(z+1)^3}\frac{(z+1-2)^2}{(z+1-1)}=\frac{1}{(z+1)^3}-\frac{1}{z}\left((z+1)^2-4(z+1)+4\right)$

i.e

$$f(z)=\frac{-1}{(z+1)^3}\left(1+(z+1)+....\right)\left(4-4(z+1)+...\right)$$
$$=-\frac{1}{(z+1)^3}\left(4-0(z+1)+...\right)=-\frac{4}{(z+1)^3}+\frac{0}{(z+1)^2}+...$$

15 Describe the behavior of the function $f(z)=\frac{z+1}{z\sin t}$ near $z=0$.

Sin

$$f(z)=\frac{-1}{(z+1)^3}\left(1+(z+1)+....\right)\left(4-4(z+1)+...\right)$$
$$=-\frac{1}{(z+1)^3}\left(4-0(z+1)+...\right)=-\frac{4}{(z+1)^3}+\frac{0}{(z+1)^2}+...$$

$$f(z)=\frac{z+1}{z\left(z-\frac{z^3}{6}+1n\right)}=\frac{1}{z^2}(z+1)\left(1+\frac{z^2}{6}-...\right)=\frac{1}{z^2}+\frac{1}{2}+\frac{1}{6}+...$$

So $f(z)$ has a double pole at $z=0$ with strength 1.

3.2.9 Groups, Rings and Fields

We recall that by a binary composition on a set S, we mean an operation that joins two elements of S to give a unique element of S. An easy example would be that of the usual addition on the set N of natural numbers. For, we know that if $a, b\in N$ then $a+b$ is 1 unique natural number suggesting thereby that addition + is a binary composition on the set N of natural numbers. There can, of course, be any number of binary compositions on a set. We now define a system to be given the name of a group.

Suppose G. Then is a non-empty set and * (star) is a binary composition on G. then G is said to form a group with respect to if

(i) Associativity. $a*(b*c)=(a*b)*c,\ \forall a, b, c, \in G$.

(ii) Existence of Identity. such that

$a\,e = e * a = a \quad \forall\, a \in G$

(iii) Existence of Inverse. For every $a \in G$, $\exists a' \in G$

(depending upon$a$), such that

$a\ * e = a' = a'\ * a = e.$

REMARKS

(a) Since * is a binary composition of G, it is understood that $\forall a, b \in G, a * b$ is a unique member of G. This property is called the closure property.

(b) If in addition to the above axioms in the definition of a group

$$a * b = b * a, \forall a, b \in G.$$

We say that G forms an ahelian group (or a commutative group). Any group, not satisfying this property is called a non-abelian or a non-commutative group.

(c) One can use any symbol for a binary composition but the most commonly used are

$*, 0, \odot, \oplus, \cdots, +$. etc.

(d) Generally, the binary composition for a group is denoted by $\odot$ dot as it makes it more convenient to write all the axioms (and they look so natural tool). So G forms a group w.r.t. the binary composition. if

(i) $a(b.c) = (a.b).c \forall a, b, c \in G$

(ii) $\exists e \in G, s.t.,\ a.e = e.a = a \forall a \in G$

(iii) $\forall a \in G,\ \exists a' \in G$ (depending upon a) s.t.,

$$a.a^{'} = a^{'}.a = e.$$

In fact, we'll drop (.) dot, too and will write ab in place of $a.b$.

(e) Whenever we say that G is a group, it will mean that there is some binary composition on G and it forms a group w.r.t. that composition [w.r.t. = with respect to].

Definitions: L group having finite number of elements is called a finite group. If it contains infinite number of elements, it is called the order of the group.

Let us now consider a few easy and simple examples.

Example 1: The set ***Z***of integers forms an abelian group w.r.t., the usual addition of integers.

Solution: Let us check all the properties in the definition of a group.

(i) **Closure:** If $a, b \in Z$ be any two integers then we know that $a + b$ is a unique integer $\Rightarrow +$ is a binary composition on Z.

(ii) **Associativity:** Let $a, b, c \in Z$ be any three elements, then

$a + (b + c) = (a + b) + c$ (4.1)

Is known to us.

(iii) Existence of Identity: We know that $\exists 0 \in Z$

s.t.,$a + 0 = 0 + a = a \forall a \in Z$

$\Rightarrow 0$is identity element in Z

(iv) Existence of Inverse: If $a \in Z$ is any member, then $\exists -a \in Z$, such that $a + (-a) = (-a) + a = 0$

$\Rightarrow -a$is inverse of a

Hence each element of Z has an inverse

(v) **Commutativity**: $a + b = b + a \forall a. b \in Z$

is also known to us.

Hence $< Z. +>$ forms an abelian group.

Example 2: The set Q of rational numbers forms an abelian group w.r.t., the addition of rational numbers.

Solution: One can easily verify all the axioms as above and thus $< Q, +>$ forms an sbelian group.

Groups, rings and fields

Example 3: $< R, +>$,where***R*** is the set of real numbers and + if the usual addition forms an abelian group.

Solution: Similar to Examples 1 and 2.

Example 4: If Q' is the set of all non-zero rational numbers and (.) denotes the usual multiplication then $< Q', . >$ forms an abelian group.

Solution: It is again easy to verify all the properties. Note I will act as identity element and 1/a as inverse of any $a \in Q^{'}$ $(a \neq 0$ given).

Example 5: Let G be the set $\{1, -1\}$. Show that this forms an abelian group w.r.t usual multiplication.

Solution: We note

$1.1 = 1 \in G$

$1.(-1) = -1 \in G$

$$(-1).(-1) = 1 \in G$$

$\Rightarrow$is multiplication is a binary composition on G i.e., closure property hodls.

Again $a^{'}.(b.c) = (a.b).c \forall a, b, c \in G$

is clear by giving any value 1 or -1 to a, b, c.

Commutativity is also obvious.

Existence of Identity. We observe.

$a.1 = 1.a = a \forall a \in G$

$1.1 = 1. = 1$

as

$(-1).1 = 1.(-1) = -1$

$\Rightarrow$ 1is identity.

Existence of Inverse. Since

$1.1 = 1. = 1$ (identity)

1is inverse of 1

Again as $(-1).1 = 1.(-1) = -1$ (identity),

$-1$is inverse of -1.

Thus each element has an inverse.

Hence $< G, . >$ is an abelian group.

Example 6: If $G = \{a\}$ and $*$ is defined on G by $a * a = a \forall a \in G$, then show that $< G, * >$ forms an abelian group.

Solution: Left to the reader as an exercise. Note that a is identity and is its own inverse.

Let us now consider some examples of such systems that do not form a group (although they are very nearly groups).

Example 7: Does the system $< Z, . >$ form a group? Where Z = set of integers and * is usual multiplication.

Solution. It is easy to see that * is a binary composition on$Z$. closure axiomholds.

Again $\quad a\,.\,(b\,.\,c) = (a\,.\,b).\,c \forall a, b, a \in Z$

$$a\,.\,b = b\,.\,a$$

So associativity and commutativity also hold.

Also since $\quad a\,.1 = 1\,.\,a = a \forall a \in Z$

1 is identity element.

Finally we note that every element of Z does not have an inverse. For instance, ∌ any integer a such that

$2\,.\,a = a\,.2 = 1$ i.e, 2 has no inverse.

$\Rightarrow < Z, : >$is not a group.

Example 8: If Q = set of rational nos., and (.) is the usual multiplication then show that $< Q, . >$ does not form a group.

Solution: If we see Example 4 we note all the properties in the definition of a group are satisfied here, except the last one as 0 has no inverse.

3.2 Some Properties

If $< G, . >$ is a group, then

(1) Identity element in G is unique

(2) Inverse of each $a \in G$ is unique

(3) $(a^{-1})^{-1} = a.\ \forall a \in G$, where a^{-1} stands for inverse of a.

(4) $(ab)^{-1} = b^{-1}a^{-1} \forall a,\ \ b \in G$

(5) Cancellation laws:

$ab = ac \Rightarrow b = c \forall a,\ \ b \in G.$

$ba = ca \Rightarrow b = c.$

Proof: (1) Suppose e and $e^{'}$ are two identity elements in G. Then, since e is identity and $e' \in G$

$e^{'}e = e^{'}e = e^{'}$.

The two taken together $\Rightarrow e = e$

Hence identity element is unique.

(2) Let $a \in G$ be some element and let x, y be two inverse elements of a in G.

Then $$ax = xa = e$$

$$ay = ya = e$$

$$x = xe$$

$$= x(ay)$$

$$= (xa)y$$

$$= ey = y.$$

Hence inverse of a is unique

We shall always denote the inverse of a by a^{-1}.

(3) Since a^{-1} is inverse of a

$$aa^{-1} = a^{-1}a = e$$

But this also implies that a is inverse of a^{-1}

$\Rightarrow a = (a^{-1})^{-1}$.

(4) We have to show that inverse of ab is $b^{-1}a^{-1}$.

we have to show that

$$(ab)(b^{-1}a^{-1}) = (b^{-1}a^{-1})(ab) = e$$

Now

$$(ab)(b^{-1}a^{-1}) = [(ab)b^{-1}]a^{-1}$$

$$= [a(b^{-1})]a^{-1}$$

$$= (ae)a^{-1}$$

$$= aa^{-1} = e.$$

And $$(b^{-1}a^{-1})(ab) = [(b^{-1}a^{-1})a]b$$

$$= [b^{-1}(a^{-1}a)]b$$

$$= [b^{-1}e]b^{-1}b = e.$$

this proves the result.

(5) Suppose $ab = ac$for some $a, b, c \in G$

we have $b = eb$

$= (a^{-1}a).b$

$= a^{-1}(ab) = a^{-1}(ac)$

$= (a^{-1}a)c = ec = c.$

Thus $ab = ac \Rightarrow b = c$

This is called *left* cancellation law.

We leave it to the reader to prove the *right* cancellation law i.e.,

$$bc = ca \Rightarrow b = c$$

We now consider a few examples.

Example 9: Show that the set Q^+ of $+ve$ rational numbers forms and abelian group *w.r.t.* the composition * defined on it by

$$a * b = \frac{ab}{2}$$

Solution Closure: Let $a,\ b \in Q^+$ be any two members,

then $a * b = \frac{ab}{2} \in Q^+$

as$\frac{ab}{2}$ will also be a $+ve$ rational number (and it is unique), thus closure holds.

Associativity: For any $a, b,\ c \in Q^+$

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = a * b = \frac{\left(\frac{\grave{a}b}{2}\right)c}{2} = \frac{abc}{4}$$

$$a * (b * c) = a * \left(\frac{cb}{2}\right) = \frac{a\left(\frac{bc}{2}\right)c}{2} = \frac{abc}{4}$$

$$\Rightarrow (a * b) * c = a * (b * c)Va, b, cc \in Q^+$$

Comutativity: $a * b = \frac{ab}{2} = \frac{ba}{2} = b * a \forall a, b \in Q^+$.

Existence of Inverse: Let $a \in Q^+$ be any element

Now a' will be inverse or a if and only if

$$a * a^{'} = a^{'} * a = 2 \text{ (identity)}$$

$$\Leftrightarrow \frac{\grave{a}a}{2} = \frac{\grave{a}a}{2} = 2$$

$$\Leftrightarrow aa^{'} = a^{'}a = 4$$

$$\Leftrightarrow a' = \frac{4}{a}.$$

Since $\frac{4}{a} \in Q^+$ whenever $a \in Q^+$. we note that each $a \in Q^+$ has inverse, namely, $\frac{4}{a}$.

Hence $< Q^+, * >$ forms an abelian group.

Example: 10: If Z is the set of integers and $*$ is a composition defined by

$$a * b = a + b + 1$$

on$Z$, where $+$ is the usual addition, then show that $< Z, * >$ is an abelian group.

Solution Closure: Let $a, b \in Z$ be any two members, then

$a * b = a + b + 1$also belongs to Z.

So closure holds.

Associativity: For any $a, b, c \in Z$,

$$(a * b) * c = (a + b + 1) * c = (a + b + 1) + c + 1$$
$$= a + a + c + 2$$
$$a * (b * c) = a * (b + c + 1) = a + (b + c + 1) + 1$$
$$= a + b + c + 2$$
$$(a * b) * c = a * (b * c) \forall a, b, c \in Z$$

Existence of identity. $e \in Z$ will be the identity

if$a * e = e * a = a \forall a \in Z$

let$a \in Z$ be any element, then

$$a * e = a + e + 1$$

Now $a + e + 1 = a \Leftrightarrow e + 1 = 0$

$$\Leftrightarrow e - 1$$

Thus -1 can act as identity.

Existence of inverse. For any $a \in Z, a'$ will be inverse of a, if

$$a * a' = a' * a = -1 \qquad \text{the identity)}$$

i.e., if $a + a' + 1 = a' + a + 1 = -1$

or if $a' = -2 - a$

Since for any $a \in Z, -2 - a \in Z$, we find that each $a \in Z$ has inverse, namely, $-2 - a$.

Commutativity. $ab = a + b + 1$

$= b + a + 1 \qquad \forall a, \in Z$

$= b * a$

Hence $< Z,* >$ is an abelian group.

Example 11: Let Q be the set of rational numbers. Define

$$G = \{(a, b) | a, b \in Q, a \neq 0\}$$

Also define a composition $*$ob G by

$$(a, b) ** (c, d) = (ac, ad + b)$$

Show that $< G,* >$ forms a non-abelian group.

Solution Closure: Let $(a, b), (c, d) \in G$ be any two elements.

Then $a \neq 0, c \neq 0$.

Now $(a, b) * (c, d) = (ac, ad + b) \in G$

as $a, c \neq 0 \Leftrightarrow ac \neq 0$.

and also ac and $ad + b$ belong to Q whenever $a, b, c, d \in Q$

Hence closure holds

Associativity. Let $(a, b), (c, e), (e, f)$ be any three members of G.

Then

$$[(a,b)*(c,d)]*(e,f) = (ac, ad+b)*(e,f)a,c,e \neq 0$$

$$= (ace, acf + ad + b)$$

$$(a,b)*[(c,d)*(e,f)] = (a,b)*(ce, cf+d)$$

$$= (ace, a(cf+d)+b)$$

$$= (ace, acf + ad + b)$$

$\Rightarrow$ Associativity holds.

Existence of inverse: We have

$$(a,b)*(1,0) = (a, a\,.\,v + b) = (a,b) = (1,0)0\ (a,b)\forall(a,b) \in G$$

$\Rightarrow$ $(1,0) \in G$is the identity.

Existence of inverse: If $(a,b) \in G$ by any element then $(c,d) \in G$ will be inverse of (a,b) if

$$(a,b)*(c,d) = (c,d)*(a,b) = (1,0)$$

Now $(a,b)*(c,d) = (1,0)$

$\Leftrightarrow (ac, ad+b) = (1,0)$

$\Leftrightarrow ac = 1$ and $ad + b = 0$

$\Leftrightarrow c = \frac{1}{a}$and $d = \frac{b}{a}$ $(a \neq 0)$

NOTE: Since $a \neq 0$, we can talk of $\frac{1}{a}$ clearly then $\left(\frac{1}{a}, -\frac{b}{a}\right)$ is the inverse of (a,b).

Hence $< G,* >$ is a abelian, consider the elements $(2,3),(1,4) \in G$.

Now $(1,2)*(1,4) = (2 \times 1, 8+3) = (2,11)$ and $(1,4)*(2,3) = (1 \times 2, 3+4) = (2,7)$

Since $(2,11) \neq (2,7)$

We note that $< G,* >$ is not abelian.

Example 12: Give an example of a system $< G,* >$ which satisfies all the axioms in the definition of a group, except the associative axiom.

Solution: Let $G = \{0, 1, 2\}$

Define a composition $*$on G, by

$$a * b = |a - b|$$

Closure: If $a, b \in G$ are any two elements (i.e., any of the elements 0,1,2)

Then $a * b = |a - b| \in G$ is clear.

Thus closure holds and so $*$ is a binary composition defined on G.

Existence of Identity: For any $a \in G$.

$$a * 0 = |a - 0| = a = |0 - a| = 0 * a$$

$\Rightarrow$ 0is the identity elements.

Existence of Inverse: Since for any $a \in G$,

$a * a = |a - a| =$
0. (identity)

It follows that a is inverse of a for any $a \in G$.

We show now that associativity does not hold in this system.

We note that

$$1 * (1 * 2) = 1 * (|1 - 2|) = 1 * 1 = |1 - 1| = 0.$$

$$(1 * 1) * 2 = |1 - 1| * 2 = 0 * 2 = |0 - 2| = 2.$$

$$\Rightarrow \; 1 * (1 * 2) \neq (* 1) * 2$$

⇒accociativity does not hold.

Example 13: Show the set $G = \{1, -1,\ i, -i\}$ forms an abelian group *w.r.t.* the usual multiplication where $i = \sqrt{-1}$.

Solution: We draw the composition table

	I	$-I$	i	$-i$
I	I	$-I$	i	$-i$
$-I$	$-I$	I	$-i$	i
i	i	$-i$	$-I$	I
$-i$	$-i$	i	I	$-I$

Suppose we want to find $i.-i$ then we locate i in the first column and $-i$ in the first row. The product $(i).(-i)$ is given in the fourth row and fifth column [fourth row is in which i lies and fifth column is in which $-i$ lies]

Closure: Holds clearly by having a look at the above table.

Associativity and Commutability are also easy to verify. (In fact the above table suggests that these properties hold).

Existence of Identity: We obverse

$$I.I = I.I = I$$

$$-I.I = I.(-I) = (-I).$$

$$i.I = I.i \quad = i$$

$$-i.I = I.(-i) = -i$$

⇒ $I$is the identity

Existence of Inverse. Since $I.I = I^{'}.I = I$ (identity)

$I$is inverse of I.

Again $(-I).(-I) = (-I).(-I) = 1$ (the ; identity)

$\Rightarrow -I$is inverse of $-I$

Also $i\,.(-i) = (-i).i = -i^2 = I$

$\Rightarrow -i$is inverse of $-i$

Thus all the elements have inverse in G

$\Rightarrow < G,. >$is an abelian group.

Example 14: Let G be the set $\{e, a, b\}$ and let a composition $*$ be defined on G by the following composition table

$*$	e	a	b	
e	e	a	b	
a	a	e	a	
b	b	b	e	

Does this system $< G,* >$ forms a group?

Solution: Closure is trivial by having a look at the composition table.

Also since $a * e = e * a = a$

$b * e = e * b = b$

$e * e = e * e = e$

$e$is the identityAgain $a * a = e = a * a \Rightarrow a$ is inverse of a

$b * b = e = b * b \Rightarrow b$is inverse of b

$e * e = e = e * e \Rightarrow e$is inverse of e.

Let us check associativity

Now $(a * b) * a = a * a = e$

But $a * (b * a) = a * b = a$

and $a \neq e$

$\Rightarrow$Associativity does not hold.

Hence $< G,* >$ doe not form a group.

Example 15: Let $G = \{0,1,2,3,4\}$. Define a binary composition $\oplus$ on G by

$$a \oplus b = c$$

where$c$ is the least non-ve remainder got by dividing $a + b$ by 5.

Show that $< G, \oplus >$ forms an abelian group.

Solution: The composition table would be

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

closure property holds by definition, since $a \oplus b$ can take values 0,1,2,3,4 only and they belong to G [also, of course, $a \oplus b$ will obviously be unique].

Commutativity is again trivial as the remainder got y dividing $a + b$by 5.

Associativity: Let $a, b, c \in G$ be any three members. We show $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

Let $a \oplus b = d$

and$(a \oplus b) \oplus c = a \oplus c = r$

now$a \oplus b = d \Rightarrow d$ is the least non-negative remainder got by dividing $a \oplus b$ by 5.

i.e.,$a + b = 5k_1 + d$

$$\begin{bmatrix} \text{for example, if } a = 3, b = 4, \text{then } a + b = 7 \\ \text{and here d} = 2, \text{k}_1 = 1, \text{so } 7 = 5, 1 + 2. \end{bmatrix}$$

Again $d \oplus c = r \Rightarrow r$ is the least non negative remainder got by dividing $d + c$by5.

i.e., $d + c5k_2 + r$ for some k_2

and, of course, $0 \leq r < 5$

Thus $a + b) + c = d + 5k_1 + c$

$$= 5k_1 + r + 5k_2$$

$$= r + 5(k_1 + k_2) = k + r$$

$$0 \leq r < 5$$

and

Hence r is the least remainder got by dividing $(a + b) + c$ by 5.

Again if $a \oplus (b \oplus c) = s$

then as above, s will be the remainder (least. non –ve) got by dividing $a + (b + c)$ by 5.

But since $(a + b) + c = a + (b + c)$

the two remainders r and s will be equal thereby proving our result.

Identity: The composition table suggests that $a \oplus 0 = 0 \oplus a = a \forall a \in G$

$\Rightarrow$ 0is identity

Inverse: W note that $0 \oplus 0 = 0 \oplus 0 = 0$

$\Rightarrow$ 0is inverse of 0

Suppose $0 \neq a \in G$ be any element then $5 - a \in G$

Also

$(5 - a) \oplus a = 0 = a \oplus (5 - a)$ (defninition)

$\Rightarrow$ $5 - a$is inverse of a $(a \neq 0)$

Hence each element has an inverse

$\Rightarrow < G_1 \oplus >$is an abelian group.

NOTE: The above composition is called addition modulo 5 and is sometimes denoted by $\oplus_5$. Thu $3 \oplus_5 2 = 0$.

One can define this composition in general on the set $(0, 1, 2, 3, \cdots, (n - 1)\}$ where it will be called addition modulo n.

Example 16: Let $G\{1,2,3,4,5,6\}$ and define a binary composition $\odot$ on G by $a \odot b = c$, where c is the least non negative remainder obtained on dividing the product ab by 7. Show that $< G, \odot >$ forms abelian group.

Solution: The composition table will be

⊙	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
6	6	5	4	3	2	1

That the closure property hold is clear from the table. Also commutativity and associativity are trivially seen to be true.

Again 1 acts as the identity element

Also $1 \odot 1 = 1 \odot 1 = 1$

$2 \odot 4 = 4 \odot 2 = 1$

$3 \odot 5 = 5 \odot 3 = 1$

$6 \odot 6 = 6 \odot 6 = 1$

$\Rightarrow$ 1is inverse of 1

2and4 are inverse of each other and so are 3 and 5.

Hence $< G, \odot >$ is an abelian group.

The above binary composition is called multiplication modulo 7 and is sometimes denoted by $\odot_7$or X_7.

The following example gives us the general result.

Example 17: (a). Let $S = \{x \in Z | I \leq x \leq n,$ where $(x, n) = I\}$ and $Z =$ set of integers and by (x, n) we mean the Highest Common Factor (HCF) of $x$and n. Define a composition $*$ on S as:

For all $a, b \in S, a * b$ is the least positive integer obtained as remainder when ab is divided by n. Show that $< S, * >$ forms an abelian group.

Solution Closure: Let $a * b = c$ for any $a, b \in S$.

Then c cannot be zero otherwise n divides ab, which is not possible as $(a, n) = 1$ and $(b, n) = 1$.

So $1 \leq c < n$.

Further if $(c, n) \neq 1$ then $\exists$ some prime number p such that it divides both c and n, which means p being a prime, either p divides ab as $a * b = c \Rightarrow ab = c + kp$ for some integer k ; p being a prime, either p divides a or p divides b.

Thus, either p divides HCF of a and n or HCF of b and n which is impossible as both $(an) = 1$ and $(b, n) = 1$.

Thus $(c, n) = 1$

$$\Rightarrow c \in S$$

$\Rightarrow$closure holds.

Associatitity: If $a * b = r_1$ and $(a * b) * c = r_2$

we get $r_1 * c = r_2$

i.e, $r_1 c = r_2 + k_1 n$ for some integer k_1 and

$$r_1 c = r_2 k_2 n$$

$\Rightarrow (ab - k_1 n)c = r_2 + k_1 n$

$\Rightarrow (ab)c = r_2 + (k_2 + k_1 c)n$

which implies that r_2 is the least non negative remainder got on dividing $(ab)c$ by n. Similarly if $a * (b * c) = r_3$, then r_3 is the least non negative integer obtained as remainder when $a(bc)$ is divided by n But $(ab)c = a(bc)$

$\Rightarrow r_2 = r_3$

$\Rightarrow (a * b) * c = a * (b * c)$

Existence of Identity: Clearly $1 \in S$ and

$$I * a = a * I = a \forall a \in S$$

$\Rightarrow I$is identity of S

Existence of inverse: Let $a \in S$, then $(a, n) = 1$

So there exist integers x and y such that

$$ax + ny = 1$$

If $1 \leq x < n$and $(, n) = 1,\ x \in S$.

If not, then by division algorithm in integers $\exists$ integers a and r such that

$$x = qn + r \qquad 0 \leq r < n$$

Now $ax + ny = 1$

$\Rightarrow aqn + ar + any = 1$

$$\Rightarrow ar = 1 + (-aq - y)n$$

So $a * r = 1$. Similarly $r * a = 1$

Again if $(r, n) \neq 1$, let p be a prime number dividing $r$and n.

Then p will divide x.

SOP divides 1 ad $ax + ny = 1$

which is absurd.

Therefore, $(r, n) = 1$, hence $r \in S$

Thus $< S,*>$ is a group.

It is easy to verify that S is abelian.

Example 17 (b). Does the set $A = \{0,1,2, \cdots, n-1\}$ with the operation of multiplication (modulo$n$) form a group for all positive integral values of$n$

Solution: The answer is no

We leave it to the reader to reason whyithas an inverse!

Example 18: Show that the set $G = \{I, w, w^2\}$ forms an abelian group under usual multiplication, where I, w, w^2 are cube roots of unity.

Solution: The composition table will be

	I	w	w^2
I	I	w	w^2
w	w	w^2	I
w^2	w^2	I	w

closure is easy to see from the above table.

I will act as the identity element

as $\quad I.I = 1 = 1.1$

$I.w = w = w.I$

$I.w^2 = w^2 = w^2.I$

$I$is clearly inverse of I.

Also as $w.w^2 = w^2.w = w^2.I$ (identity).

We note that w is inverse of w^2

andw^2 is inverse of w.

Asscoativity can be verified by the help of the composition table, commutativity being obvious.

Hence the result.

Example 19: Show that the set of all 2×2 matrices over integers forms an abelian group w.r.t. matrix addition.

Solution: Let $M = \left\{\begin{pmatrix} at \\ cd \end{pmatrix} | a, b, c, d \in Z\right\}$

Then clearly M is a non empty set.

Also-since for any two matrices $AB \in M, A + b$ is given a 2×2 matrix belonging to M.

We note that closure holds.

Again the fact that matrix addition is both commutative and associative proves the other two axioms in the definition of a group.

The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ will act as the identity and for any matrix

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the matrix $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ will act as inverse.

Hence the set M forms an abelian group.

[Render is referred to the Chapter on Matrices for the proofs of the above results].

Example 20: Prove that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ form a group under matrix multiplication.

Solution: Let G be the set containing the matrices.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we note that

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A$$

$$II = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$AI = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A$$

$$AA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

So closure holds.

Iacts as the identity element.

Also A and I act as their own inverse.

Associativity can also be checked easily.

Hence the set G forms a group under matrix multiplication. [It will be and abelian group.]

REMARK: Compare this with Example 5.

Definition If $< G, . >$ is a group and $a \in G$, then we denote $a \,.\, a$ by a^2 which is again a member of G. Similarly by a^3, we mean $a^2.a$. This notation can be extended further also. One can show that

$$a^m a^n = a^{m+n}$$
$$(a^m)^n = a^{mn}$$
$$a^{-m} = (a^m)^{-1} \text{etc.}$$

Example 21: If G is a group, such that $(ab)^2 = a^2b^2 \forall a, b \in G$, show that G is abelian.

Solution: Let $a, b \in G$ be any two members

Then $(ab)^2 = a^2b^2$ (given)

$\Rightarrow (ab)(ab) = (aa)(bb)$ (Definition)

$\Rightarrow (aba)b = (aab), b$ (Associativity)

$\Rightarrow aba = aab$ (Cancellation laws)

$\Rightarrow G$is abelian

Example 22: If every elements of a group G is its own inverse then G is abelian.

Solution: Let $a, b \in G$ be any two elements

$\Rightarrow ab \in G$(closure)

$\Rightarrow ab = (ab)^{-1}$ (given)

$\Rightarrow ab = b^{-1}a^{-1}$

But as $a, b \in G, a^{-1} = a, b^{-1} = b$

Thus $ab = ba \forall a, b \in G$

$\Rightarrow G$is abelian

Example 23: A group G having three elements if always abelian.

(C.A., November, 1974)

Solution: Let G have three elements e, a, b such that $e \neq a \neq b$ [e being the identity].

Now $a, b \in \Rightarrow ab \in G$ (closure)

Thus either $a\,b = e$ or $ab = a$ or $ab = b$

Suppose $ab = a$

$\Rightarrow \quad ab = ae$

$\Rightarrow \quad b = e$ (Cancellation law)

But this is not true. So $ab \neq a$

Suppose $ab = b$

$\Rightarrow \quad ab = eb$

$\Rightarrow \quad a = e$

Which is not true. so $ab \neq b$

Hence $ab = e \Rightarrow b$ is inverse of a

i.e., $b = a^{-1}$

Thus $ab = aa^{-1} = e = a^{-1}a = ba$.

Also clearly $ea = ae, eb = be, ee = ee$ and so each element commutes with all others.

$\Rightarrow G$ is abelian.

Example 24: Show that for given a , b in a group G the equations $ax = b$

$ya = b$

have unique solutions for x and y in G .

solution: We show that the equation $ax = b$ has a unique solution of G, i.e., $\exists$ a unique value of x, in G which satisfies this equation.

Now, $ax = b \quad \Rightarrow a^{-1}(ax) = a^{-1}b$

$\Rightarrow (a^{-1}a)x = a^{-1}b$

$\Rightarrow \quad ex = a^{-1}b$ where e is identity element of G

$\Rightarrow \quad x = a^{-1}b$.

Now, $a^{-1}b \in G$ as $a, b \in G$, and this is the value of x that satisfies the equation $ax = b$.

Suppose now $x = x_1$ and $x = x_2$ are two solutions of the equation $ax = b$

Then $\quad ax_1 = b$

$ax_2 = b$

$\Rightarrow ax_1 = ax_2$

$\Rightarrow x_1 = x_2$ by cancellation laws

$\Rightarrow$ solution is unique.

Similarly it can be shown that $y = ba^{-1}$ is the unique solution of the equation $ya = b$.

EXERCISES

1. If Q^+ = set of + ve rational numbers and * be a composition defined on Q^+ by

   $$a * b = \frac{ab}{3}$$

   Show that $< Q^+ ,* >$ forms an abelian group.

2. Let $G =$ set of all real number except -1. Define * on G by $a * b = +bab \forall a, b \in G$. Show that $< Q^+ ,* >$, is an abelian group.

3. Prove that the set Q of all rational numbers other than 1 with the operation * defined

CHAPTER 4

4.1.1 FOURIER SERIES AND INTEGRAL TRANSFORMS

A Fourier series is a representation employed to express a periodic function $f(x)$ defined in a interval say $-\pi,\pi$. Fourier series are sines and cosines and therefore if will be useful to state these properties of $\sin nx$ and $cos\, nx$ where n is an integer that make them continuous in interval $-\infty < x < \infty$ or within other given range for instance: A function $f(x)$ is continuous at $x = x_0$ if $f(x_0^+) = f(x_0^-) = f(0)$

$$\text{where } f(x_0^+) = \lim_{x \to x_0} f(x) \text{ is the lim as } x \to x_0$$

From the values $x > x_0$ and $f(x_0^-) = \lim_{x \to x_0} f(x)$ is the limit as $x > x_0$.In a simple term, a function is continuous at every point a $a < x < b$.

One major characteristic of Fourier series is its applicability in representation of a periodic function as already mentioned earlier. A function $f(x)$ is periodic if it is defined for all real x and if there is some positive number say p which is independent of x such that $f(x + p) = f(x)$ for all x. This number, p is called a period of $f(x) = f(x + 2)p$ On other hand a periodic function is a function that is repetitive in cycles of equal duration eg are $e.m.f, a.c$ current etc.

$$f(x) = a_0 + a_1 Cos\, x + a_2 Cos 2x \ldots + a_n\, Cos\, nx +$$

$$\ldots + b_1\, Sin\, x + b_2 Sin 2x + b_n Sin\, nx$$

$= a_0 \sum_{n=1}^{\infty} a_n Cos\, nx + \sum_{n=1}^{\infty} b_n Sin\, nx (4.1)$

One should note that Fourier series expansion and the determination of Fourier constant is valid under the following assumptions;

i. The expansion of $f(x)$ in a series of *sines* and *consines* of integral multiples of x is possible in the give interval.
ii. The given function $f(x)$ is single-valued continuous and integrable in the given ranges
iii. The series $a_0 \sum_{n=1}^{\infty} (a_n Cos\, nx + b_n Sin\, nx)^{\ldots(2)}$ is term by term integrable. That is the series is uniformly convergent in the interval $(-\pi, \pi)$.

iv. The function $f(x)$ satisfies Dirichlet

Let us suppose that $f(x)$ is a periodic as already stated with period 2π as given in equation (2).

Given such a function $f(x)$, we want to determine the coefficient a_n and b_n in the series of equation (2).

We first determine a_0.integrating term by on both sides of equation (2) from $-\pi$ to π we have

$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}[a_0 + \sum_{n=1}^{\infty}(a_n Cos\ nx + b_n sin\ nx)]\ dx$ (4.2)

If term by term integration of the series is allowed, then we obtain.

$\int_{-\pi}^{\pi} f(x)dx \neq \int_{-\pi}^{\pi}\left[a_0 dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n Cos\ dx + b_n \int Sin\ nx\ dx\right]$ (4.3)

The first term on the right hand side equals $2\pi a_0$

All the other integrals on the right seen by performing the integrations hence our first result is

$$a_0 = \frac{1}{-\pi}\int_{-\pi}^{\pi} f(x)dx \quad \ldots.\ldots \qquad (4.4)$$

We now determine $a_1, a_2 \ldots$ by a similar procedure. We multiply equation (2)by $Cos\ nx$, where in is any fixed positive integer, and then integrate from $-\pi$ to π.

Thus,

$$\int_{-\pi}^{\pi} f(x) Cos\ mx\ dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n Cos\ nx + b_n Sin\ nx\right] Cos\ mx\ dx$$

$$a_0 \int_{-\pi}^{\pi} Cos\ mx\ dx = \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n Co\ nx\ Cos\ mxdx + \int_{-\pi}^{\pi} b_n Sin\ nx\ Cos\ mx\ dx.$$

The first integral $\int_{-\pi}^{\pi} a_n Cos\ nx = 0$

Where $\int_{-\pi}^{\pi} nx\ Cos\ mxdx = \frac{1}{2}$

When $n = m$, the surviving term is

$$\frac{1}{2}\int_{-\pi}^{\pi} Cos\ (n-m)x\ dx = \frac{1}{2}\int_{-\pi}^{\pi} dx = \frac{1}{2}2\pi = \pi$$

This implies that

$$\int_{-\pi}^{\pi} f(x) cos\ xdx = a_n\ \pi;\ \text{Then}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) cos\ mx\ dx \ldots\ldots \qquad (4.5)$$

Also, to determine b_n, we multiply equation (3) by $Sin\ mx$

$$\int_{-\pi}^{\pi} f(x) Sin\ mx\ dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n Cos\ nx + b_n Sin\ nx\right] Sin\ mx\ dx$$

$$= \int_{-\pi}^{\pi} a_0 Sin\ mx\ dx = + \sum_{n=1}^{\infty} [a_n \int_{-\pi}^{\pi} Cos\ \ nx Sin\ mxdx + b_n \int_{-\pi}^{\pi} Sin\ nx\ Sin\ mxdx.$$

The first integral is zero. The next integral is of the type considered before which is known to be zero for $n = 1{,}2$ etc. For the last integral we have.

$$\int_{-\pi}^{\pi} Sin\ \ nx Sin\ mxdx = \frac{1}{2}\int_{-\pi}^{\pi} Cos(n-m)x\ dx \int_{-\pi}^{\pi} Cos\ (n+m)dx$$

The last term is zero the first term on the right is zero when $n \neq m$ and is π when $n = m$.

$$\therefore \int_{-\pi}^{\pi} Sin\ nx\ Sin\ mx\ dx = \pi$$

Substituting in equation 5 we obtain that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Sin\ mx\ dx \ldots \qquad (4.6)$$

The following are known as the Euler formulae

$$\left.\begin{aligned} a_o &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx && (a) \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Cos\ nx\ dx && (b) \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Sin\ nx\ dx && (c) \end{aligned}\right\} (4.7)$$

Work Example

1. Find the Fourier series of

$$f(x) = \begin{cases} 1 \text{ for } -\frac{\pi}{2} < x < {}^{\pi}/_{2} \\ -1 \text{for } -\frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

Solution

In this solution, we use the Euler formulae to obtain the Fourier Co-efficient a_o, a_n and b_n.

$$a_o = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} f(x)dx + \int_{\pi/2}^{3\pi/2} f(x)dx$$

Where $f(x) = 1$ and -1 respectively

$$a_o = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1dx + \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}(-1)dx = \frac{1}{2\pi}[(\pi) - (\pi)] = 0$$

From $7(b)$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(x)Cos\,nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} f(x)Cos\,nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} Cos\,nx\,dx + \int_{\pi/2}^{3\pi/2} -Cos\,nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{Sin\,nx}{n}\right]_{-\frac{\pi}{2}}^{\pi/2} + \frac{1}{\pi}\left[\frac{-Sin\,nx}{n}\right]_{\pi/2}^{3\pi/2}$$

For odd n,

$$a_n = \frac{2}{n\pi} + \frac{2}{n\pi} = \frac{4}{n\pi}$$

$$a_n = \frac{4(-1)^n}{n\pi}$$

For $n = 0$ and even, $a_n = 0$

$$\therefore a_n = \begin{cases} 0 \quad \text{for } n = 0 \text{ or even} \\ \dfrac{4(-1)^n}{n\pi} \text{ for } n = 1 \text{ or odd} \end{cases}$$

To evaluate b_n;

$$b_m = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(x)Sin\,nx\,dx$$

$$b_n = \int_{-\pi/2}^{\pi/2} Sin\,nx\,dx + \int_{\pi/2}^{3\pi/2} -Sin\,nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{-Cos\,nx}{n}\right]_{-\frac{\pi}{2}}^{\pi/2} + \frac{1}{\pi}\left[\frac{-Cos\,nx}{n}\right]_{\pi/2}^{3\pi/2}$$

$$b_n = 0$$

The series $f(x)$ is as given below
$f(x) = a_0 + a_1 \, Cos\, x + a_2 Cos\, 2x \ldots b_1 Sin\, 2x + b, Sin\, 3x$

Where a_0 and $b_1 \ldots b_n$ are zero,

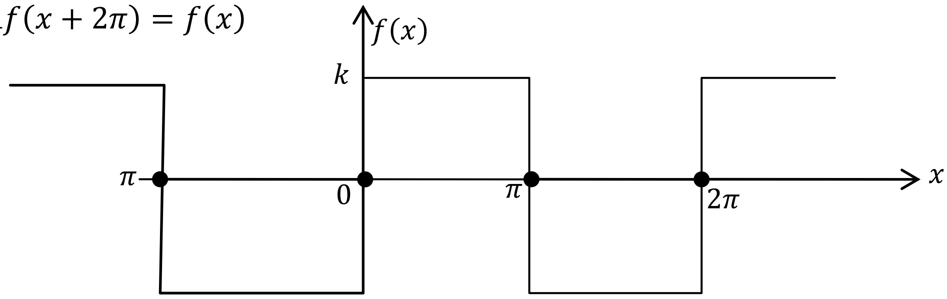
$$f(x) = \frac{4(-1)^n}{n\pi} Cos\, nx = \frac{4}{\pi}\left[Cos\, x - \frac{Cos\, 3x}{3} + \frac{Cos\, 5x}{5} - \frac{Cos\, 7x}{7} + \cdots\right]$$

Example 2

Find the Fourier coefficient of the periodic function

$$f(x) = \begin{cases} -k \text{ when } -\pi < x < \pi \\ k \text{ when } 0 < x < \pi \end{cases}$$

and $f(x + 2\pi) = f(x)$



This is the graph of the given function.

Solution

We use Euler formula

$$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} -kdx + \frac{1}{2\pi}\int_{0}^{\pi} kdx$$

$$a_o = \frac{1}{2\pi}[-kx]_{-\pi}^{0} + \frac{1}{2\pi}[kx]_{0}^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Cos\, nx\, dx = \frac{1}{\pi}\int_{-\pi}^{0} -kCos\, nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} kCos\, nx\, dx$$

$$a_n = \frac{1}{\pi}\left[\frac{-Sin\, nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{kSin\, nx}{n}\right]_{0}^{-\pi} = 0 + 0$$

$$a_n = 0$$

Similarly

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Sin\, nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} -k\, Sin\, nx\, dx + \int_{0}^{\pi} k\, Sin\, nx\, dx\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{kCos\, nx}{n}\Big|_{-\pi}^{0} \frac{-kCos\, nx}{n}\Big|_{0}^{\pi}\right.$$

$$= \frac{k}{n\pi}[Cos\, 0 - Cos(-n\pi) - Cos\, n\pi + Cos\, 0]$$

$$\frac{k}{n\pi}[2Cos\, 0 - 2Cos\, n\pi] = \frac{2k}{n\pi}[1 - Cos\, n\pi]$$

For $n = 1,3, \ldots$ odd, $Cos\, n\pi = -1$ but for $n = 2$ or even, $Cos\, n\pi = 1$

$$\therefore 1 - Cos\, n\pi = \begin{cases} 2 & \text{for } n \text{ add} \\ 0 \text{ for } n \text{ even} \end{cases}$$

This implies that b_n for even $n = 0$

Thus, the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{n}, b_3 = \frac{4k}{3n}, b_5 = \frac{4k}{5n}$$

Since a_0 and $a_n = 0$, the corresponding Fourier series is

$$f(x) = \frac{4k}{n}\left[Sin\, x + \frac{1}{3}Sin\, 3x + \frac{1}{5}Sin\, 5x + \cdots\right]$$

4.1.2 Fourier Transform

They are of the following forms.A Fourier since transforms can be subdivided lim two namely the finite Fourier since and transform of the infinite Fourier since transform $A; i$ the infinite Fourier since transform of a function $f(x)$ of x such that $0 < x < \infty$ is denoted by $f_S(n)$, being a positive integer and is defined as

$f_S(n) = \int_0^\infty f(x) \sin nxdx$ (4.8)

$f(x)$is called the inverse Fourier sin transform of $f_S(n) and\ is\ defined\ as$

$f(x) = \frac{2}{\pi}\int_0^\infty f_S(n)\ Sin\ \ nxdx$ (4.9)

Thus if$f_S(n) = f_S[F(x)]$then$F(x) = F_S^{-1}[f_S(n)]$

Where f is the symbol for Fourier transform and F^{-1} for its inverse.

Example 1: Find the inverse since transform of $e^{-\lambda n}$

Solution

$$F_S^{-1}\left[e^{-\lambda n}\right] = \frac{2}{\pi}\int_0^{\infty} e^{-\lambda n}\ Sin\ \ nxdx = \frac{2}{\pi}\left[\frac{e^{-\lambda n}}{\lambda^2 + x^2}(-\lambda\, Sin\, nx - s\cos nx\right]$$

$$= \frac{2}{\pi}\frac{x}{\lambda^2 + x^2}$$

a. The finite Fourier since transform of a function $f(x)$ of x such that $0 < x < l$ is denoted by$f_S(n), n$ being a positive integer and it defined as

$$f_S(n) = \int_0^{\infty} f_S(n)\ Sin\ \frac{n\pi x}{l} Qx$$

In a case where $l = \pi$, the is becomes

$$f_S(n) = \int_0^{\infty} f_S(n)\ Sin\ \ nxdx \quad (4.10a)$$

And the inversion Formula is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} f(x)\sin\ nxdx \qquad (4.10b)$$

Whence is the coefficient of $\sin nx$ in the expansion of $f(x)$ in a since series and is given by

$$a_n = \frac{2}{\pi}\sum_{n=1}^{\infty} f(x)\sin\ nxdx = \frac{2}{\pi} f_S(n) \qquad (4.11)$$

Example 2: Find the Fourier since transform of $f(x) = x$ such that $0 < x < 2$

Solution: We have $f_S(n) = \int_0^2 f()x \sin\frac{n\pi x}{2} dx : l = 2$ in this case,

$$f_S(n) = \int_0^2 x\ Sin\ \frac{n\pi x}{2} dx = \left[x - \cos\frac{n\pi x}{2}\right]_0^2 + \int_0^2 \frac{2}{n\pi}\cos\frac{n\pi x}{2} dx$$

$$= \frac{4}{n\pi}\cos n\pi$$

b. Fourier cosine transform: They are also subdivided into two farts infinite and finite.

i. The infinite Fourier cosine transform of $f(x)$ for $0 < x < \infty$ is defined as

$f_S(n) = \int_0^2 f(x)\ \cos nxdx\ \ n$ being a positive integer.

Here the function $f(x)$ is called the inverse cosine transform of $F_C(n)$ and is defined as

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(n)\cos nxdx$$

Thus if $f_C[f(x)]$, then $f(x) = f_C^{-1}[f_C(n)]$

Example 3: Find the cosine transform of $x^n e^{-ax}$

Solution

Recall that $\int_0 \; f(x)\cos nxdx \; = \int_0^\infty e^{-ax}\cos nxdx$

Thus $f_C(n) = \int_0^\infty x^n e^{-ax}\cos nxdx$

We differential $x^n e^{-ax}$ n times with respect to a we now find that

$$\int_0^\infty x^n e^{-ax}\cos nxdx = (-1)^n \frac{d^n}{da^n}\frac{a}{a^2+n^2}$$

$$\therefore f_C(n) = in\cos\frac{\{(n+1)tan^{-1}\,{}^n/_a\}}{(a^2+n^2)^{(n+1)}/as}$$

Example 4: Find$F_C^{-1}\{e^{-\lambda n}\}$

$$F_C^{-1}[e^{-\lambda n}] = \frac{2}{\pi}\int_0^\infty e^{-\lambda n}\cos nxdx = \frac{2}{\pi}\left[\frac{e^{-\lambda n}}{\lambda^2+x^2}(-\lambda\cos nx + x\sin nx)\right]_0^\infty$$

$$\frac{2}{\lambda^2+x^2}$$

ii. The finite Fourier cosine transform of $f(x)$ of $0 < x < l$is defined as

$f_C(n) = \int_0^1 f(x)\cos\frac{n\pi x}{l}\; dx$when$l = \pi$, this becomes
$f_C(n)\int_0^1 f(x)\cos\; nxdx$ and the inversion formula is

$$f(x) = \frac{1}{\pi}f_C(0) + \frac{2}{\pi}\sum_{n=1}^{\infty} f_C(n)\cos nx$$

When $f_C(0) = \int_0^\infty f(x)dx$

Example 5: Find the finite Fourier cosine transform of x

Already $f_C(n) = \int_0^\pi x\cos nxdx = \left.\frac{x\sin\; nx}{n}\right|_0^\pi - \frac{1}{n}\int_0^\pi \sin xdx$

$$0 - \frac{1}{n}\left[\frac{\cos\; nx}{n}\right]_0^\pi = \frac{1}{n^2}[(-n)^n - 1],$$

However, if $n = 0 f_C(n) = \int_0^{\pi} x dx \ = \left[\frac{x^2}{2}\right]\begin{matrix}\pi\\0\end{matrix} = \frac{\pi^2}{2}$

Another important of Fourier transforms is complex Fourier transforms of a function.

It is defined as $f(n) = \int_{-\infty}^{\infty} F(x)e^{inx} \ dx$

Where e^{inx} is said to be the kernel of the transform. The inversion formula is $f(x) = \frac{1}{2\pi} n$

The Fourier transforms. Provides a representation of function defined over an infinite interval with no particular periodicity in terms of a super position of sinusoidal functions. It may thus be considered as a generalization of the Fourier series representation of periodic function.

It is one of the most important of Fourier transforms as we shall see in the theory of transforms.

Theory of Fourier transformation in the Fourier series case the function C_n is called the Fourier coefficient or spectrum of $f(x)$

Where

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{i(n\pi x/L)} \qquad (4.12)$$

$$(-\infty << x << \infty),$$

Where

$$c_n = \int_{-l}^{l} f(x)e^{-i(n\pi x/l}dx$$

C_n =We set$^{n\pi}/_{l} = k$; then$\Delta k = {}^{\pi}/_{l} \ \ \Delta n$.

The adjacent values of k are obtained by putting $\Delta n = 1$, which corresponds to ${}^{l}/_{\pi} \ \ \Delta k = 1$

Then multiplying each term of the Fourier sieves by ${}^{l}/_{\pi} \ \ \Delta k$ and writes

$$f(x) = \sum_{n=-\infty}^{+\infty} \left({}^{l}/_{\pi}\, C_n\right) \Delta k\, l \qquad (4.13)$$

Where

$$\frac{l}{\pi}\, Cn = \frac{1}{2\pi}\int_{-<}^{<} f(x)e^{-i(n\pi x/<)}dx$$

Going to K −notation and writing

$\left(\frac{1}{\pi}\right) C_n = C_l(k)$,we can obtain $C_l(k) = \frac{1}{2\pi}\int_{-l}^{l} f(x)e^{-ikx}\,dx$ and

$$f(x) = \sum_{<:k/_{n=-\infty}}^{+\infty} l(k)e^{ikx} \quad \Delta k. \qquad (4.14)$$

If we now het $l \to \infty$,we expect then second sum to go over into an integral and we obtain ∞.

$$C(k) = \lim_{l \to \infty} C_l(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

Is known as Fourier transform while

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \qquad (4.15)$$

The inverse transforms. We note that his concept is analogous to the concept of Laplace transform. Just as we can also observe that the only difference between direct and inverse Fourier transform is the sign in exponential function.

The first example we wish to discuss here is form the Fourier transform of Gaussian probability function.

$$f(x) = Ne^{-\propto x2} \quad (N, \propto = \text{constant})$$

In this case, its Fourier transform $f(k)$ will be denoted by

$T[f(x)]$and will be calculated from

$$F(k) = T[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx$$

$$= \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\propto x^2}e^{ikx}\,dx \qquad (4.16)$$

The technique used for the calculation of this integral is to be dealt with below

$$T[f(x)] = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(-\propto x^2 + ikx)}\,dx$$

We simplify $= -\propto x^2 + ikx$ by completing the square:

$$-\propto^2 x^2 + ikx \Rightarrow \left(x\sqrt{\propto} - i\,{}^{k}/_{2\sqrt{\propto}}\right)^2 - \frac{k^2}{4\propto}$$

And make the change of variables.

Let $U = x\sqrt{\propto} - i\,{}^{k}/_{2\sqrt{\propto}}$ implies that

$$dU = \sqrt{\propto}\ dx$$

Substituting back into $T[f(x)]$

$$T[f(x)] = \frac{N}{\sqrt{2\pi}} e^{-k/4\propto} \int_{-\infty}^{\infty} e^{-u}\, du = \sqrt[N]{\frac{1}{2\propto}}\, e^{-\frac{k^2}{4\propto}}$$

The most obvious interesting fact here is that it is seen that $f(k)$ or $T[f(x)]$ is also a Gaussian probability function with a peak at the

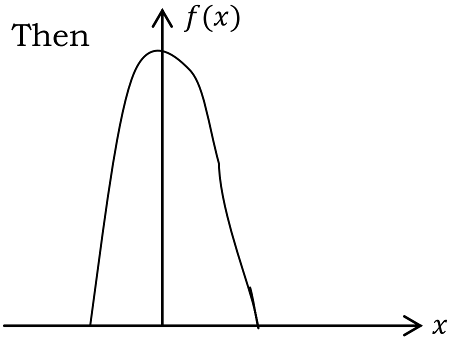
origin monotone decreasing as $k \to \pm\infty$ it is also interesting that if $f(x)$is sharply peaked (Large x), $T[f(x)]$is flattened.

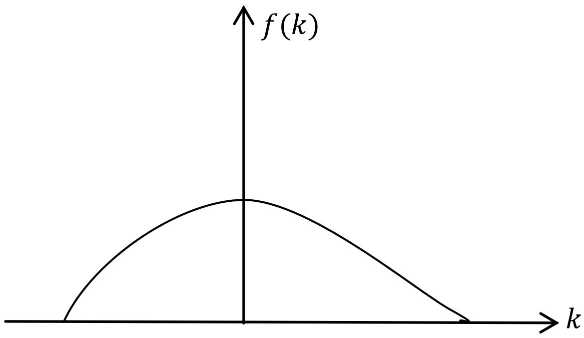
Here $f(x)$ is sharply peaked as a result after the transform $T[f(x)]$ is flattened. Similarly if $f(x)$ is flattened, transform $T[f(x)]$ is peaked.

Also if $f(t)$ is single square pulse with a height say h and with b centred on the orgine of t such as a top−hat function

$$f(t) = h\ \left(-\frac{b}{2} < + < \frac{b}{2}\right) \text{with } f(t) = 0 \text{ else where}$$

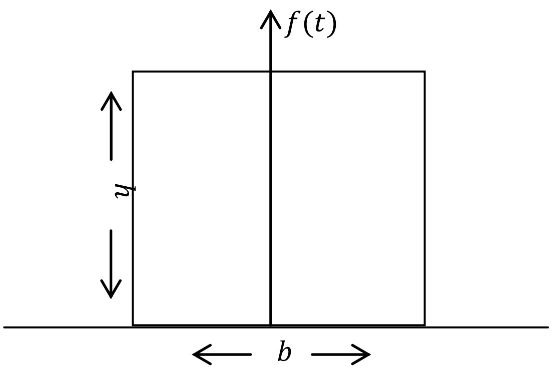
Then

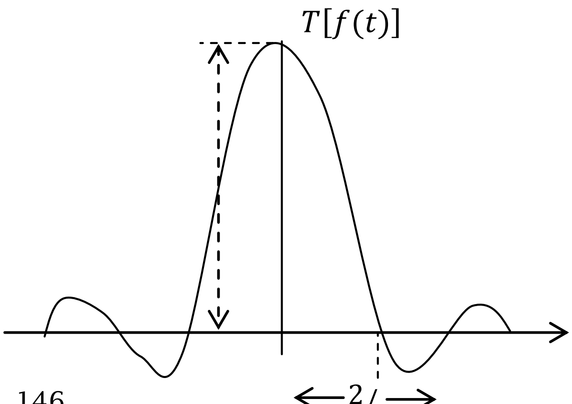




$$T[f(x)] = \int_{-\infty}^{\infty} f(t)\exp(-2\pi e\ vt)dx$$

Gives the transform which will be interpreted diagrammatically as





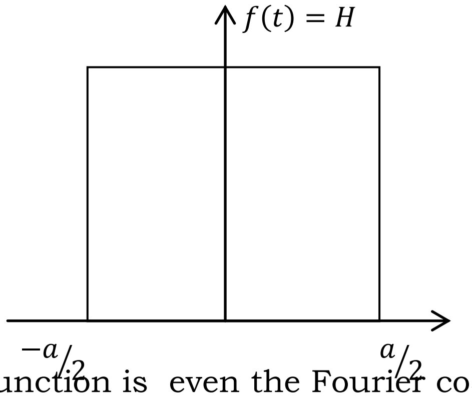
V

Example:

Find the Fourier transform of the function given by $f(x) = H$, and $-{}^{a}/_{2} < x < {}^{a}/_{2}$

Solution

It can be represented as



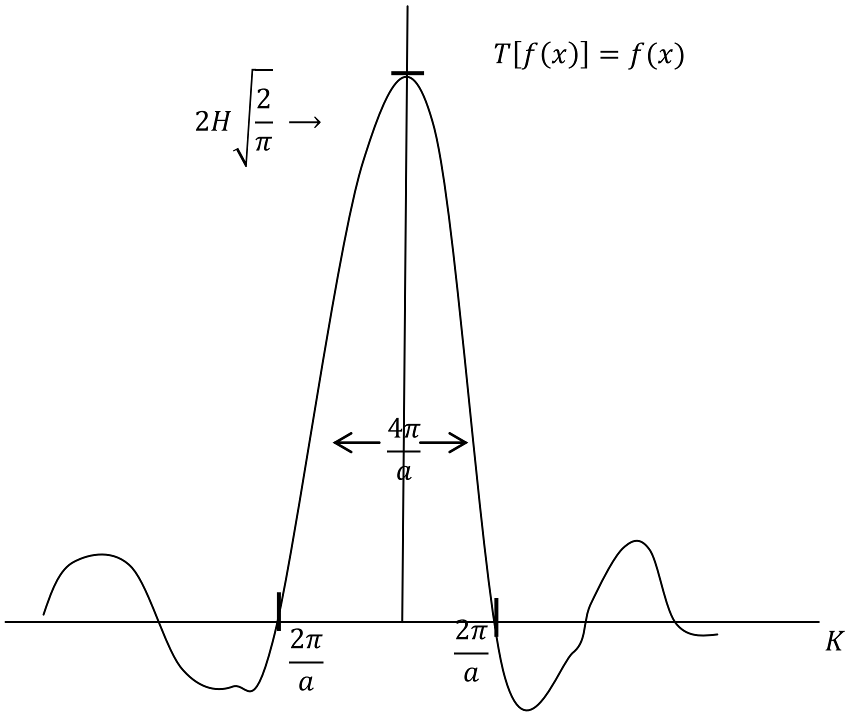
The function is even the Fourier cosine transform

$$T[f(x)] = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x) \cos kx dx$$

$$= \frac{2}{\sqrt{\pi}} \int_{\frac{-a}{2}}^{a/2} H \cos Kx dx = H \sqrt{\frac{2}{\pi}} \left[\frac{\sin \ kx}{k} \right]_{-a/2}^{a/2}$$

$$= 2H \sqrt{\frac{2}{\pi}} \frac{\sin a/2}{k} = \sqrt{\frac{8H^2}{\pi}} \frac{\sin a/2}{k}$$

This can be sketched as shown below;



4.1.3Laplace transform

Laplace transform is one the method used in solving differential equations especially the second-order differential equations with constant coefficients such as

$$a\frac{d^2}{dx^2} + b\frac{dy}{dx} + cy = F(x)$$

The main advantage of this method is that if involves algebraic processes with involvement of the initial conditions from the early stage of the problem. Another important advantage of Laplace transform is that enables one to deal with situations where the function is discontinuous unlike that of the Fourier transforms.

Laplace transform $T[F(x)] = T(P) \equiv \int_0^\infty f(x)e^{-Px}dx$

Provided that the integral exist, in practice, for a given function $f(x)$ there will be some real number P_0 such that the integral in the define equation exist for $P > P_0$, but diverges for $P \leq P_0$

By direction application of the definition of Laplace transform, we find that (i). For $f(x) = 1$

$$T[f(x)] = T[1] = \int_0^\infty e^{-Px} I\, dx = -\frac{I}{P}e^{-Px}\int_o^\infty = \frac{-I}{P}[0-1] = \frac{I}{P}$$

(ii). For $f(x) = x$

$$T[f(x)] = T(x) \equiv \int_0^\infty xe^{-Px}dx$$

The solution here involves integration by part.

$$\Rightarrow \int_0^\infty xe^{-Px}dx = -\frac{1}{P}xe^{-Px}\int_0^\infty -\frac{1}{P^2}xe^{-Px}\int_0^\infty = -0 + \left[0 - \left(-\frac{1}{P^2}\right)\right] = \frac{1}{P^2}$$

(iii). For $f(x) = x^2$,

$$T[f(x)] = T(x^2) = \int_0^\infty x^2e^{-Px}dx$$

$$= -\frac{1}{P}x^2e^{-Px}\int_0^\infty + \frac{1}{P}\int_0^\infty 2xe^{-Px}dx$$

Since as we can see that in (ii) that $T[x] = \frac{1}{P^2}$

Then, the last term become $\frac{1}{P}\cdot\frac{2}{P^2}$

Where the first term is zero

$$\therefore T(x^2) = \frac{2}{P^3}$$

In general it can be shown that $T(x^n)$ for $f(x) = x^n$ is given as

$$\frac{n!}{P^{n+1}}$$

(iv). For $f(x) = e^{ax}$

$$T[f(x)] = T[e^{ax}] = \int_0^\infty e^{ax}\, e^{-Px}dx$$

$$= -\frac{1}{P-a}e^{-(P-a)x}\int_0^\infty = -\frac{1}{P-a}(0-1) = -\frac{1}{P-a}$$

(v). for trigonometrically function. Say $\cos ax + i\sin ax = e^{iax}$

$$\therefore T[\cos ax + i\sin ax] = T\left[e^{iax}\right] = \int_0^\infty e^{iax}\, e^{-px}dx$$

$$\equiv \frac{1}{P - ia}$$

In this case, we rationalize thus

$$\frac{1}{P-ia}\times\frac{P+ia}{P+ia}=\frac{P+ia}{P^2+a^2}$$

In this case, we equal real part to real and complex to complex to complex

i.e $T[\cos ax + i\sin ax] = \frac{P}{P^2+a^2} + \frac{ia}{P^2+a^2}$

$$\Rightarrow T[\cos ax] = \frac{P}{P^2+a^2} \; and$$

$$i\sin ax = \frac{ia}{P^2+a^2} \Rightarrow [\sin ax] = \frac{a}{P^2+a^2}$$

Example 1 obtain the Laplace transform of $f(x) = 3\cos 4x + 4e^{5x}$

$$T[3\cos 4x + 4e^{5x}] = 3T[\cos 4x] + 4T[e^{5x}]$$

$$= \frac{3P}{P^2+16} + \frac{4}{P-5}$$

Example 2

Use Laplace transform to solve the initial value problem given as

$$y' + 2y = 2x : y(o) = 5/2$$

Solution

We take the transform of the differential equation

$$T[y'] + 2T[y] \neq 2T[x]$$

$$T[y'] + 2T[y] = 2T[x]$$

$$T[y'] = PT[y] - y(o)$$

$$\therefore PT[y] - y(o) + 2T[y] = \frac{2}{P^2}$$

$$(P+2)T[y] - y(o) = \frac{2}{P^2}$$

The boundary condition specifies that $y(o) = 5/2$

$$:(P+2)T[y] = \frac{5}{2} + \frac{2}{P^2}$$

$$\therefore T[y] = \left[\frac{5}{2} + \frac{2}{P^2}\frac{1}{(P+2)}\right]$$

$$= \frac{5}{(P+2)} + \frac{2}{P^2(P+2)}$$

Here we have to resolve the second term at the right-hand side into partial fraction, $\frac{2}{P^2(P+2)} \equiv \frac{P}{A} + \frac{B}{P^2} + \frac{C}{(P+2)}$

$$2 \equiv AP(P+2) + B(P+2) + CP^2$$

$$2 = (A+C)P^2 + (2A+B)P2B$$

$$2 = 2B; B = 1$$

$$2A + B = O; 2A = -B; 2A = -1$$

$$A + C = 0; C = -A$$

$$C = \frac{1}{2}$$

$$\Rightarrow T[y] = \frac{5}{2(P+2)} + \frac{2}{P^2(P+2)}$$

$$= \frac{5}{2}\frac{1}{P+2} - \frac{1}{2}\frac{1}{P} + \frac{1}{P^2} + \frac{1}{2}\frac{1}{P+2}$$

$$= 3\frac{1}{P+2} + \frac{1}{P^2} + \frac{1}{2}\frac{1}{P}$$

The inverse transforms are

$$3\frac{1}{P+2}3e^{-2x}, \frac{1}{P^2} = x, \frac{1}{2}\frac{1}{P} = \frac{1}{2}$$

$$\therefore y = 3e^{-2x} + x - \frac{1}{2}$$

Example 3: Solve the initial value problem using Laplace transform

$$y''' + 4y' = 10e^x \; where$$

$$y(o) = 3, y'(o) = 4, y''(o) = 10$$

Solution, we take Laplace transform of the differential equation.

$$T[y'''] + 4T[y'] = 10T[e^x].$$

$$4\{T[y] - y(o)]$$

Now

$$T[y'''] = PT[y''] - y''_{(o)}10T[e^x].$$

Again $PT[y''] = P\left[PT[y'] - y'_{(o)}\right]$

$$PT[y'] = P\left\{[PT[y] - y_{(o)}\right\}$$

Now we substitute back along with the initial conditions, to obtain

$$P^3T[y] - 3P^2 - 4P10 + T[y] - 4y_{(o)}]$$

$$= \frac{10}{P+1}$$

$$\Rightarrow (P^3 + 4P)T[y] - 3P^2 - 4P - 12 - 10 = \frac{10}{P+1}$$

$$T[y] = \frac{3P^2}{P^3 + 4P} + \frac{4P}{P^3 + 4P} + \frac{22}{P^3 + 4P} + \frac{10}{(P+1)(P^3 + 4P)}$$

$$= \frac{3}{P^2 + 4} + \frac{4}{P^2 + 4} + \frac{22}{P(P^2 + 4)} + \frac{10}{P(P+4)(P-1)}$$

We simplify by solving all the into partial fraction where

$$T[y] = \frac{3P}{P^2 + 4} + \frac{4}{P^2 + 4} + \frac{11}{2}\frac{1}{P} + \frac{2}{P-1} + \frac{11}{2}\frac{1}{P^2 + 4} - \frac{5}{2P}\frac{1}{P}$$

After this we take the inverse transform: Thus

$T[y] = \cos 2x - \frac{7}{2}\sin 2x + 2e^{x} + 3$

Exercise 4.0

(1). Solve $y^{''} - 4y = 5\cos x$ using Laplace transform given that

$$y_{(o)} = -2, y^{'}_{(o)} = 10$$

$$Ans: T[y] = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x} - \frac{7}{2}\cos x.$$

(2). Solve $y^{''} + 5y^{'} + 6y = 4x$

$$at\ x = 0, y = 0, and\ y^{'} = 0$$

$$Ans: y = \frac{2}{3}x - \frac{5}{9} + e^{-2x} - \frac{4}{9}e^{-3x}$$

(3). Solve $y^{''} - 2y^{'} + 10y = e^{2x}$

$$y_{(o)} = 0, y^{'}_{(o)} = 1$$

$$Ans: y = \frac{1}{10}[e^{2x} - e^{x}\cos 3x + 3e^{x} sin\ 3x]$$

4 Given the function $f(x) = x^2$, obtain the values for Euler formula and form the Fourier series.

5. Show that the Fourier series of the function $f(t)$ whose period is 4 is

$$k - \frac{2k}{\pi}\left[Sin\ t - \frac{1}{3}Sin\ 2t + \frac{Sin\ 3t}{3} - \frac{Sin\ 4t}{4} + \cdots\right]$$

If

$$f(t) = \begin{cases} 0 \text{ when} - 2\pi \le t \le -\pi \\ k\text{when} - \pi \le t \le \pi \\ 0 \text{ when } \pi \le t \le 2\pi \end{cases}$$

6. Find the Fourier coefficients in the interval $0 \le x \le 2\pi$ of the periodic function defined as

$$f(x) = x\ for - \pi \le x \le \pi \text{and} f(x + 2\pi) = f(x).$$

$$Ans:\ f(x) = 2\left[Sin\ x - \frac{Sin\ 2x}{2} + \frac{Sin\ 3x}{3} - \frac{Sin\ 4x}{4} \pm \cdots\right]$$

7.Find the Fourier series of the function defined as

$$f(x) = \begin{cases} 0 & \text{for} - \pi < x < 0 \\ 1 & \text{for } 0 \leq x \leq \pi \end{cases}$$

$$Ans\text{: } f(x) = \frac{1}{2} 4 \frac{2}{\pi}\left[Sin\ 2x + \frac{Sin\ 3x}{3} + \frac{Sin\ 5x}{5} - \frac{Sin\ 7x}{7} + \cdots\right]$$

CHAPTER 5

5.1.0 Ordinary differential equation

Differential equations are those groups of equations that contain derivatives. These are variety of differential equations.

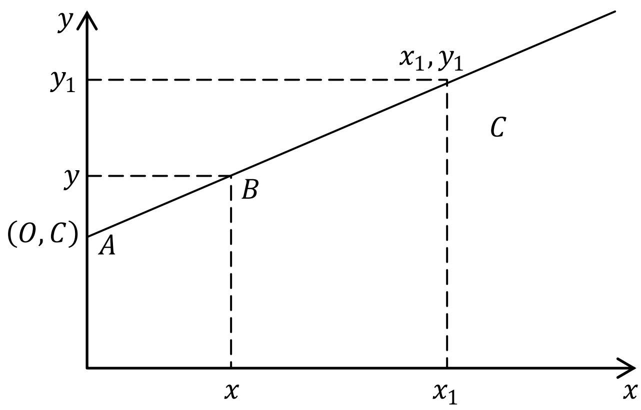
1) Ordinary differential equation which contain ordinary derivatives that describes the relationship between these derivatives of the dependent variable, usually called y with respect to the independent variable, usually known as y.

2) Partial differential equations describe the relationship between a dependent variable say v and other two or more independent variables say $(x, y, t\ etc)$ and partial differential coefficients of v with respect to these independent variables.

They will be treated respectively in this chapter.

section 1: formation of ordinary linear differential equations. Ordinary differential equation can be formed from ordinary simple algebraic expressions; order concept relation motions such vibrations and scientific concepts.

For instance, from differential equation that describes the gradient of straight lime shown below.



An important point to note here is that the gradient of straight line is constant every point considered on the straight-line.

Thus,

$$\frac{y-c}{x=0} = \frac{y_1 - y}{x_1 - x} = \frac{dy}{dx}$$

$$\Longrightarrow x\frac{dy}{dx} = y = c$$

$$\therefore y = x\frac{dy}{dx} + c \cdots\cdots \quad (6$$

Another Example

Curve is defined by the condition that the sum of the x and y intercept of its tangent is always equal 2. Express the condition by the means of differential equation.

Solution

Let the tangent to the curve be $y = mx + c$, where $ml = dy/dx$, the intercept on $x - axis$ is when$y = 0x = -c/m$.

Intercept on $y - axis$ is when $x = 0y = c$.

$$\therefore \left[c + \left(1 - \frac{c}{m}\right)\right] = 2 \cdots$$

$$mc - c = 2m$$

$$c(m - 1) = 2m \cdots \cdots$$

From $y - mx + c$

$$c = y - mx \qquad (6.2)$$

Substitute in to form c

$$(y - xm)(m - 1) = 2m$$

$$ym - yxm^2 + xm - 2m$$

$$\Longrightarrow (y + x - 2)\frac{dy}{dx} - y = 0 \cdots \qquad (6.3)$$

Example 3

If current at any time, flowing through conductor is i, form an expression describing the rate of flow of current.

$$\frac{di}{dt} = ki \cdots$$

Where k is constant.

At this point it is important observe that all the differential coefficients are of first order and first degree except the second one that contains second degree.

The order of differential equation is the number of differential equation is the number of differentiation i.e$\frac{dy}{dx}$ first order; $\frac{d^2y}{dx^2}$, second order while the degree is the power to which differential coefficient is raised.

$\left(\frac{dy}{dx}\right)^2$, Second degree; $\left(\frac{dy}{dx}\right)$ First degree

Example 4 Form differential equation from the equation given below

$$x = e^{2t}(A + Bt) \cdots \quad (a)$$

Differential equation as obtained from primitive expression is known as general solution.

The solution is a function containing some arbitrary constant.

Solution

We differentiate successively twice equation (a)

$$\frac{dx}{dt} = 2e^{2t}(A + Bt) + e^{2t}B \cdots \qquad (b)$$

$$\frac{d^2x}{dt^2} = 4e^{2t}(A + Bt) + 4Be^{2t}B \cdots \qquad (c)$$

Substituting equation a and b.

$$\frac{dx}{dt} = 2x + Be^{2x} \cdots \quad (d)$$

Multiply equation $(d) \times 4$

$$4\frac{dx}{dt} = 8x + 4Be^{2x} \cdots \quad (e)$$

Similarly substituting equation (a) in (c)

$$\frac{d^2x}{dt^2} = 4x + 4Be^{2t} \cdots \qquad (f)$$

Equation $e\ to$ equation f

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0$$

Trial question,

Given $y = Ae^{2x} + Be^{-x}$, show that the differential equation with that solution is

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 2y = 0$$

Example 5:

An object of mass m falls from rest under gravity subject to a air resistance proportional to its velocity for the differential equation for the concept.

$$ma - mg \propto -v$$

$$m\frac{d^2x}{dt^2} - mg = cv.$$

$$\frac{d^2x}{dt} - g = -\frac{c}{m}\frac{dx}{dt}.$$

Thus the correct differential equation is

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} - g = 0 \cdots$$

Example 6:

Newton's law of cooling states that the rate of cooling decreases of temperature of a hot body is proportional to excess of temperature of the body over that of the surroundings. Using t for time and $\theta^0 c$ for temperature in 0_C and $\%0_C$ for temperature of the surrounding (assume constant), express the low in the form of differential equation.

$$\frac{d\theta}{d\theta} \propto (\theta - \theta_o)$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_o)$$

Example 7:

A body moves in a straight-line so that when it is $x\,cm$ from a point 0 on the lime its acceleration is 9 $cm\,s^{-2}$ towards 0.

Write down the differential equation describing the motion

Solution

$$a \propto -x$$

$$\frac{d^2x}{dt^2} = -9x$$

This can also be written as$v\frac{dv}{dx} = -9x$

Similarly if a mass m is applied to one end of a suspended spring whose other end is fixed, it will produce the extension x, which according to Hooke's law is proportional to the applied force

$F = mg$.

Thus,

$$F = Kx = Mg.$$

where$K$ is stiffness of the spring and g is acceleration due to gravity, If additional force is applied, which produces extension y; the tensional force, $T = K(x + y)$ while the resultant downward force $F = Mg - K(x + y)$. For the combined action of the spring and gravity as it is known that

$$Mg = -Kx$$

Then $f + \text{key} = 0$

$$\Rightarrow M\frac{d^2y}{dx^2} + kg = 0 \cdots$$

This equation is differential equation for force Oscillations of undamped mass-spring system. If the system is placed in a resistive medium in which the resistive force is $-v$ or ry'

By Newton's Law,

$$my'' + ry'ky = 0$$

becomes the Oscillatory motion.

It should be mentioned here that all these equations as formed are known as homogeneous linear differential equation.

The equation becomes inhomogeneous if a periodic forcing term is introduced say $T(t)$

For instance if in equation above we introduce a force term (T(t) such that it becomes

$$my'' + ry' + ky = T(t).$$

From the demonstrations as shown above in some of the derived differential equations it is apparent that differential equations are form from an idea or primitive equations.

5.1.3 Solution of First Order Linear Differential Equation.

The solution first order linear differential equation depends on the nature and form of the equation.

5.2.1Direct Integration

One of the methods of solution is by direct integration method such as simple equation as $(1\, PCos\, 2\theta)\frac{dy}{d\theta} = 0$ can be solved by direct integration

$$(1 + Cos\, 2\theta)\frac{dy}{d\theta} = 2$$

$$dy = \frac{2d\theta}{1 + Cos\, 2\theta}$$

$$1 + Cos\, 2\theta = Cos^2\theta$$

$$\int dy = \int \frac{2d\theta}{Cos\theta} = \frac{2d\theta}{Sec^2\theta}$$

$$y = tan\theta + c$$

Example 1

Solve the equation

$$dT/dt = k(T_O^4 - T^4)$$

Where k and T_O are constants.

Solution

We solve by integration

$$\int \frac{dT}{(T_O^4 - T^4)} = \int k\, dt$$

We simplify $T_O^4 - T^4 = (T_O - T)(T_O - T)(T_O^2 - T^2)$ and resolve it using partial fraction

$$\int \frac{1\ \ dT}{4T_O^3(T_O - T)} + \int 4T_O^3(T_O - T) + \int \frac{T_O^2 dT}{2(T_O^2 - T^2)} = \int k\, dt$$

$$\frac{1}{4T_O^3}\ln\frac{1}{(T_O - T)} + \frac{1}{4T_O^3}\ln\frac{1}{(T_O - T)} + 2\tan^{-1}\left(\frac{T}{T_O}\right) = kt + c$$

$$\Rightarrow \frac{1}{4T_O^3}\left(\frac{T_O + T}{T_O - T}\right) + 2\tan^{-1}\left(\frac{T}{T_O}\right) = kt + c.$$

5.2.2 Separation of variable

Another method used in solution of first order linear differential equation is by using separation of variable.

Example 2

Solve$\frac{dy}{dx} = x(y-2)$

Solution

In this case, we separate the variable first as thus,

$$\frac{dy}{(y-2)} = xdx$$

and integrate directly.

$$\int \frac{dy}{(y-2)} = \int xd$$

$$\ln(y-2) = \frac{1}{2}x^2 + c$$

In a case where the equation is not separable, the solution takes a different solution as will be discussed subsequently.

In a case when

$$M = a_1x + b_2y + c_2 \text{and}$$

$$N = a_2x + b_2y + c_2$$

Where m, n are linear function of x and y.

That is $Mdx + Ndy = 0$

There are 2 cases here

Case I: when

$a_1b_2 + a_2b_1 = 0$, then the transformation $V = a_1x + b_1y$, reduces the equation to separable equation.

Case II: If $a_1b_2 \pm a_2b_1 \neq 0$, we proceed to the following manner:

Solve $a_1x + b_1y + c_1 = 0$ and

$a_2x + b_1y + c_2 = 0$ Simultaneously to obtain $x = h, y = k$ and then transform

$$x = x' + h, \qquad y = y + k$$

this reduces the equation to homogeneous equation.

Example 3:

Solve the equation $(x+2y+1)+(3x+6y-4)dy=0$

We test for case I: where $a_1=1, a_2=3$ i.e $a_1b_2-a_2b_1$

$$(1)(6)-(3)(2)=0$$

Therefore, $V=x+2y$

Then $y=\frac{v-x}{2}$

$$dy=\frac{dv-dx}{2}$$

We substitute in the original equation

$$(v+1)dx+(3v-4)\frac{1}{2}(dv-dx)$$

$$\Rightarrow (6-v)dx+(3v-4)dv=0$$

This is now separable.

It becomes

$$dx=\left(\frac{3v-4}{6-v}\right)dv$$

$$\Rightarrow x=3v+14\ln\left(\frac{1}{6-v}\right)+c$$

Example 4:

Solve the equation $dy/dx=\frac{x-y+2}{x+y}$, reducing it to a homogeneous equation by the change of variables $x=X-1, y\,Y+a,$ [Note that this implies a change of origin to $(-1,1)$ the point of intersection of a straight line $x-y+2=0$and $x+2=0$]

The new axes are parable to the old one so that

$$\frac{dy}{dx}=\frac{dy}{dx}]$$

Solution

$$\frac{dy}{dx}=\frac{X-1-Y-1+2}{X-1+Y+1}$$

$$\frac{X-y}{X+y}$$

Let $Y = UX$

So that

$$\frac{dy}{dx} = U + X\frac{dy}{dx}$$

$$\Rightarrow U + X\frac{dy}{dx} = \frac{X(1-u)}{X(1+U)}$$

$$X\frac{dU}{dX} = \frac{1-2U-U^2}{1+U}$$

$$\int \frac{(1+U)}{1-2U-U^2} = \int \frac{dX}{X}$$

To solve the integral on the left hand side,

Let $t = 1 - 2u - u^2$

$$\frac{dt}{du} = -2(1+u)$$

$$-\frac{1}{2}\int_{t(1+u)}^{(1+u)} dt = -\frac{1}{2}\ln t$$

Substituting back, we obtain

$$-\frac{1}{2}\ln(1-2u-u^2) = \ln X$$

$$\ln(1-2u-u^2)^{-1} = 2\ln X = \ln X^2$$

$$\ln(1-2u-u^2)\,X^2 = C$$

Recall that $y = ux$

$$u = \frac{y}{x}$$

$$\left(1 - \frac{2y}{X} - \frac{y^2}{X^2}\right)X^2 = C$$

$$\frac{(X^2 - 2YX = Y^2)}{X^2}X^2 = C$$

$$(x+1)^2 - 2(x+1)(y+1)^2 = c$$

$$x^2 - y^2 - 2xy + 4x + 1 = c$$

5.2.3 Exact Differential Equations

An exact first-degree first-order differential equation is of the form

$M(x, y)dx + N(x, y)dy = 0$and for which is necessary condition for the equation to be exact;

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{and } \frac{\partial^2 M}{\partial y \partial x} = \frac{\partial^2 N}{\partial x \partial y}. \qquad (6.4)$$

Example 5:

Check whether or not

$$(3x^2 + 6xy^2 - 6y^2)dx + \left(6x_y^2 + 12xy + 2y - 1\right)dy = 0$$

Is exact and hence solve it.

Solution

Let $M = 3x^2 + 6xy^2 - 6y^2$ *and*

$$N = 6x^2y - 6xy^2 - 12xy + 2y - 1$$

$$\frac{\partial M}{\partial y} = 12xy - 12 \text{ and } \frac{\partial N}{\partial x} = 12xy - 12$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$this fulfills the condition for exactness

Hence to solve it,

$$\int M\, dx \text{ and } \int N\, dy$$

$$\int (3x^2 + 6xy^2 - 6y^2)\, dx = 3x^2 + 3x^2y^2 - 6y^2x$$

$$\int (6x^2y - 12x = 2y - 1\,)\, dy = 3x^2y^2 + x^3 - y^2 - y$$

After the integration we now compare the two $x^3 + 3x^2y^2 - 6y^2x - y^2 - y + c$

$$f(x, y) = x^3 + 3x^2y^2 - 6xy^2 - y^2 - y + c$$

If the equation is not exact, we first all have to determine a factor that will beeither usedto multiply or divide the equation in order to make it exact. This factor is known as integrating factor. This factor has no general method of finding it, it is often by inspection. As earlier discussed in the case of exact, it is not exact when

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad (6.5)$$

In this case, a factor says $M(x, y)$ that makes it possible to obtain the condition such that is what we

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Refer to as integrating factor.

For instance, if the differential equation is of the form.

$$\frac{dy}{dx} + P(x)y + Q(x).$$

The integrating fact is obtain thus

$$\int Pdx$$

After this we multiply both sides with the integrating factor

$$\underset{e}{\int Pdx}\frac{dy}{dx} + \underset{e}{\int Pdx}\; P(x)y + \underset{e}{\int Pdx}\; Q(x).$$

Example 6:

Solve the differential equation

$$\frac{dy}{dx} + 6y = \cos x$$

Solution

The integrating factor is $e^{\int Pdx}$

Where $P = 6$

$$If = e^{\int 6dx} = e^{6x}$$

$$\therefore \frac{d}{dx}(e^{6x}y) = e^{6x}\cos x$$

$$ye^{6x} = \int e^{6x}\cos x\, dx$$

Now if we integrate the right-hand side, the problem is solved.

$$ye^{6x} = \frac{1}{7}[\cos x - \sin x]$$

$$y = \frac{e^{6x}}{7}(\cos x - \sin x)$$

Example 7:

Solve

$$\frac{dy}{dx} - y\tan x = 3e^{\sin x}$$

Solution

Here $P = \tan x,$

Thus if$= e^{\int \tan x dx} = e^{\ln \cos x} = \cos x$

$$\therefore \cos x \frac{dy}{dx} - \cos x \tan x = 3\cos x\, e^{\sin x}$$

$$y\cos x = 3\int \cos x\, e^{\sin x} dx$$

$$y\cos x = 3e^{\sin x} dx + c$$

$$y = \frac{1}{\cos x} = \left(3e^{\sin x} dx + c\right)$$

5.2.4 Initial and boundary conditions

As regards any differential equation whether first or second order, the solution always involves arbitrary constant or arbitrary function in particular case unless if there are additional information given about the variables of contained in the equation.

These extra information are called the initial conditions or boundary conditions

There are three bread classes of boundary conditions

a) Dirichlet boundary conditions: The value of the dependent variable is specified on the boundary
b) Neumann boundary conditions: The normal derivative of the dependent variable is specified on the boundary.
c) Cauchy boundary conditions: Both the value and the normal derivative of the dependent variable are specified on the boundary.

Cauchy boundary conditions are analogous to the initial conditions for a second order ordinary differential equation. These are given at one end of the interval.
For instance, if in example 6 the condition that room temperature is 20^0c and that it takes a particular body 12 minutes to cool from 100^0c to 50^0c, find the time taken by the body to cool from 50^0c to 25^0c
Now side

$$\frac{d\theta}{dt} = k(\theta - \theta_0)$$
$$\int \frac{d\theta}{(\theta - \theta_0)} = -k \int dt$$
$$\ln(\theta - \theta_0) = -kt + c$$
$$(\theta - \theta_0) = e^{-kt} e^{c} = ce^{-kt}$$
$$\theta = \theta_0 + ce^{-kt}$$

Now the boundary condition says that when $\theta_0 = 20^0 c$

$$\theta = 20 + ce^{-kt}$$

When $\theta = 100c, t = 0$

$$\vdots\; 100 = 20 + ce^{o}$$
$$\therefore c = 80^0 c$$
$$\Longrightarrow \theta = 20 + 80\, e^{-kt}$$

When $\theta = 50, t = 12\, minutes$

$$\therefore 50 = 20 + 80\, e^{-12k}$$
$$k = \frac{\ln 3/8}{-12} = 0.081736$$
$$\theta = 20 + 80\, e^{-0.08176\, t}$$

The given boundary conditions have enable us arrive at the definite value.
Now for the body to cool from 50^0c to 25^0c we substitute in the equation

$$25 = 20 + 80\, e^{-0.08176\, t}$$
$$\frac{5}{80} = 80\, e^{-0.08176\, t}$$
$$t = \frac{\ln 3/80}{0.081736} = 33.92126\, minutes$$

Example 8:
An electric circuit consists of an inductance of $0.1H$ a resistance of $200l$ and an $e_0 mf$ of $200v$. Find the current i in terms of t if $i = 4$ when $t = 0$.
Solution
The equation $L\frac{di}{dt} + iR = E$ is first order differential equation it can be written as

$$\frac{di}{dt} + \frac{iR}{L} F\, {}^{E}\!/_{L}.$$

Being inexact, differential equation we obtain the integrating factor.

$$e^{\int Pdx} = e^{\int R/Ldt} = e^{\int {}^{Rt}\!/_{L}}$$

Substituting

$$e^{Rt/L}\frac{di}{dt}+\frac{iR}{L}e^{Rt/L}=E/L\,e^{Rt/L}$$

$$\frac{d}{dt}\left(ie^{Rt/L}\right)=E/L\,e^{Rt/L}$$

$$ie^{Rt/L}=E/L\int e^{Rt/L}\,dt=E/\not{L}^{Rt/L}+k$$

$i=E/R+k_1e^{-Rt/L}$general solution

Now the values are

$$R=200L, E=200V, L=0.1H$$

$$i=10+k_1e^{-2t}$$

When $i=4, t=0$ are the boundary conditions?

$$\therefore 4=10+k_ie^0$$

$$k_1=-6$$

$$\therefore i=10-6e^{-2t}.$$

Example 9:

A bacterial population B is known to have a rate of growth proportional to B itself. If between noon and 2pm, the population triples, at what time, no control being exerted, should B because 100 times what it was at noon.

Solution

$$\frac{dB}{dt}\propto B$$

In this case, there is no negative because

$$\frac{dB}{dt}\propto kB$$

$$\int\frac{dB}{B}=\int k\,dt$$

$$\ln B=kt+C$$

$$B=e^{kt+C}$$

$$B=Ae^{kt}$$

$$t=2\ hour$$

$$B$$

If it is tripled,

$$B=3A$$

$$\therefore 2A=Ae^{kt}$$

$$K=\frac{\ln 3}{2}=0.549306144$$

$$B=Ae^{0.549306144}$$

When $B=100A$

$$100A=Ae^{0.549306144\,t}$$

$$t=8.3836\ hour$$

5.2.5 Bernoulli Equation

This equation is of the from

$$\frac{dy}{dx}+P(x)y=y^nQ(x).$$

This type of equation needs transformation before it can be solved

The transformation is done by dividing through by y^n

$$\Rightarrow y^{-n}\frac{dy}{dx}+y^{(1-n)}P(x)=Q(x).$$

With this transformation, the equation will be made linear and can be solved by obtaining the integrating factor.
Now to solve this,
Let $y^{1-n} = u$
Thus; $\frac{du}{dy} = (1-n)y^{(1-n)}\frac{dy}{dx}$

$$\therefore y^{-n}\frac{dy}{dx} = (\frac{1}{1-n})\frac{du}{dx}$$

Substituting,

$$\frac{1}{1-n}\frac{du}{dx} + (1-n)P(x)U = (1-n)Q(x).$$

Example 10:
Solve $y^1 - 2y = y^5x$.
Solution
First divide through by y^5

$$y^{-5}y^1 - 2y^{-4} = x.$$

Let $u = y^{-4}$

$$\frac{du}{dx} = -4y^{-5}\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{4}\frac{du}{dx}$$

$$\Longrightarrow -\frac{1}{4}\frac{du}{dx} - 2u - x.$$

$$\frac{du}{dx} - 4(-2) = -4x.$$

$$\frac{du}{dx} + 8u = -4x$$

Now find the integrating factor, if;

$$if = e^{\int 8dx} = e^{8x}$$

$$\Longrightarrow e^{8x}\frac{du}{dx} + 8ue^{8x} = 4xe^{8x}$$

$$\therefore ue^{8x} = -4\int xe^{8x}dx$$

$$u = \frac{-x}{2} + \frac{1}{16} + ce^{-8x}$$

Recall that $u = y^{-4}$.

$$\therefore y^{-4} = \frac{-x}{2} + \frac{1}{16} + ce^{-8x}$$

Example 11:
Solve

$$\frac{dy}{dx} - 2y = y^3(\cos 3x - \sin 3x)$$

Solution
1st transform

$$y^{-3}\frac{dy}{dx} - 2y = y^{-2} = (\cos 3x - \sin 3x)$$

Let $u = y^{-2}$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$\Rightarrow y^{-3}\frac{dy}{dx}=\frac{1}{2}\frac{du}{dx}$$

Substituting, we get

$$-\frac{1}{2}\frac{du}{dx}-2u=\cos 2x-\sin 2x$$

Which now linear

$$\frac{du}{dx}+4u=-2(\cos 3x-\sin 3x)$$

With $P(x)$

$$\therefore if=e^{\int 4dx}=e^{4x}$$

$$e^{4x}\frac{du}{dx}+4ue^{4x}=-2e^{4x}(\cos 3x-\sin 3x)$$

$$\frac{du}{dx}(ue^{4x})=2e^{4x}(-\cos 3x+\sin 3x)$$

$$Ue^{4x}=+2\int -e^{4x}\cos 3x\,dx+\int e^{4x}\sin 3x\,dx$$

The right-hand side is of a standard integral which can easily be integrated using integration by part.

$$Ue^{4x}=2\left(\frac{1}{4^2+3^2}\right)[-4e^{4x}\cos 3x+3e^{4x}\sin 3x]+C$$

$$U=\frac{-8}{25}\cos 3x+\frac{6}{25}\sin 3x+Ce^{-4x}$$

$$y^{-2}=\frac{-8}{25}\cos 3x+\frac{6}{25}\sin 3x+Ce^{-4x}$$

5.3.1 Higher order differential equations.

This is linear equation which is of the form.

$$P_n(x)\frac{d^ny}{dx^n}+P_n(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots P_2(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+=Q(x).$$

Where $P_n^{\cdot}, P_{n-1}, P_2^{\cdot}, P_1, P_0^{\cdot}$ and Q $are\ functions\ of\ x.$

In this case, it will be referred as linear differential equations with variable co-efficient e.g. Legendre equation and Bessel equation etc.

If on the other hand, $P_n, P_{n-1}, P_2, P_1, P_0$ are constant, the linear differential equation will be referred as linear equation with constant coefficients. The approach to solution of these two outlined form of differential equation are different slightly but in other aspect the same in that they involve.

First of all,we find the general solution of the complementary equation. That is the equation formed by setting

$$Q(x)=0$$

For that with variable coefficients, we must find n linear independent $P_0(x)y$ functions that satisfy it. As soon this is obtained, the general solution of given a linear superposition of these n functions. With n^{th} solutions

$y_1()x= c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x).$

Where c_n are arbitrary constants that may be determined if the n boundary conditions are given.

The linear combination $y(x)$ is known as complementary function of the linear differential equation.

Now is it possible to establish that any n indicial solution to the equation is linearly independent. For this to be so over an interval, there must exist any set of constants $c_1, c_2 \cdots c_0$ such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0.$$

Which can only occur at such a case when $c_1 = c_2, = c_3 \cdots = c_n = 0$.
This can be confirmed by repeatedly differentiating the above equation $n-1$ time in order to obtain n simultaneous equations for $c_1 = c_2 \cdots c_n$

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0$$
$$c_1 y_1'(x) + c_2 y_2'(x) + \cdots + c_n y_n'(x) = 0$$
$$\vdots$$
$$\vdots$$
$$c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \cdots + c_n y_{n2}^{(n-1)}(x) = 0$$

If the determinant of the co-efficient of $c_1, c_2 \cdots c_n$ is non-zero, then the only solution to the differential equation is trivial solution $c_1 = c_2^{\cdots} = c_n = 0$
i.e

$$W(y_1, .. y_n) = \begin{vmatrix} y_1 & yc_2 \cdots & y_n \\ & y_1' & \\ & y_1' & \\ y_1^{(n-1)} & & y_1^{(n-1)} \end{vmatrix} \neq 0$$

Are linearly independent over the interval
$W(y_1 .. y_n)$ is known as the Wronskian of the set of function.

5.3.2 The Wronskian

Suppose we have two solution $U_1(x)$ and $U_2(x)$ of the differential equation $\frac{d^2u}{dx^2} + P_{(x)} \frac{du}{dx} + 9_{(x)} U = 0$.
Then the Wronskian $w.r.t\ A$ is defined by $W(x) = U_1(x) \frac{du}{dx} 2^{(x)} - U_2^{(x)} \frac{du_1(x)}{dx}$

$$= \begin{vmatrix} U_1 U_2 \\ U_1' U_2' \end{vmatrix} = B$$

Theorem;
U_1 and U_2 are linearly dependent if and only if $W = 0$ otherwise they are linearly independent.
Proof.
For if they are linearly dependent, there exist two constant λ_1 and λ_2 not both zero such that

$$\lambda_1 U_1(x) + \lambda_2 U_2(x) = 0.$$

The condition here is that (C) and (D) must have non-trivial solution for λ_1 and λ_2. That is both λ_1 and λ_2 must not be zero.

$$\Rightarrow W = \begin{vmatrix} U_1 U_2 \\ \\ U_1' U_2' \end{vmatrix} = 0$$

$$U' U_1 U_2' - U_2 U_1' = 0$$

But $\frac{d}{dx}\left[\frac{U_2^{(x)}}{U_1^{(x)}}\right] = \frac{U_1 U_2' - U_1 U_2'}{U_1^2}$

If $U_1 U_2' - U_2 U_1' = 0$ since $U_1 \neq 0$

Hence $W = 0$

$\Rightarrow \frac{d}{dx}\left[\frac{U_2}{U_1}\right] = 0 \Rightarrow \frac{U_2}{U_1} = constant$.i.e$U_2$ and U_1 are linearly dependent

$\Rightarrow \lambda_1$andλ_2 are non-zero

The D.E satisfied by Wronskian as U_1 and U_2 are two solutions of A, then

$$U_1'' + PU_1' + qU_1 = 0 \cdot\cdot E$$
$$U_2'' + PU_2' + qU_2 = 0 \cdot\cdot F$$

Now if we multiply E by $U_2$and F by U_1 and subtract, we obtain

$$U_2 U_1'' - U_2 U_2'' + P(U_2 U_1' - U_2 U_2') = 0 \cdot\cdot (G)$$

Since $W(x) = U^1 U_2' - U_2 U_1'$

Thus $\frac{du}{dx} U_1 U_2'' + U_1' U_2'' - U_2 U_1'' - U_2' U_1' \equiv U_1 U_2'' - U_2 U_1''$

Hence (G) is nothing but

$$\frac{dw(x)}{dx} + P(x)W(x) = 0 \cdots (H)$$

which is the differential equation obeyed by the Wronskian.

This equation H can be written in the form can be written in the form $\frac{1}{w}\frac{dw}{dx} = -P(x) \cdots (I)$ of which if we integrate both sides

$$\ln W(x) = -\int_{x_1}^{x_1} P(x')\, dx' + C$$

$$W(x) = exp[-\int_{x_1}^{x_1} P(x')\, dx' + C_0$$

$$= W(x_1) exp[-\int_{x_1}^{x} P(x')\, dx'$$

$$\therefore W(x) = W(x_1) exp[-\int_{x_1}^{x} P(x')\, dx' \cdots (J)$$

Now we carry out solution in terms of U_1 and$W$.

Since $\frac{d}{dx}\left[\frac{U_2}{U_1}\right] = \frac{U_1 U_2' - U_2 U_1'}{U_1^2} \equiv \frac{W}{U_1^2}$

Then $U_1^2 \frac{d}{dx}\left[\frac{U_2}{U_1}\right] = W = (K)$

i.e$\frac{d}{dx}\left[\frac{U_2}{U_1}\right] = \frac{W}{U_1^2}$ which after integration becomes

$$U_2(x) = U_1(x) \int_{x_1}^{x} dx' \frac{W(x')}{U_1^2(x)} \cdots (L)$$

Where $W(x)$ is given by (J)

The summary is that if one solution of equation (A) is known then the second solution of (A) is given by (L).

If on the other hand, we have two linearly independent solutions $U_1(x)$ and $U_2(x)$ of the same equation A, we can construct from it a solution.

$$U_3(x) = aU_1(x) + bU_2(x) \cdots (\mathrm{M})$$

Which has an arbitrary assigned value and the derivative at same point x_0.Suppose at x_0, we want.

$U_3(x) = \propto_0$ and $U_3^1(x) = \mathrm{B}_0$,then we choose a and b so that

$$\left.\begin{aligned} \propto_0 \equiv U_3(x_0) = aU^{'(x_0)} + bU_2(x_0) \\ B_0 \equiv U_3^1(x_0) = aU^{'(x_0)} + bU_2^1(x_0) \end{aligned}\right\} (N)$$

In this case since the Wronskian does not vanish for linearly independent solution as earlier explained we can solve by Crammer's rule for a and b

Linear differential equations with constant coefficients

$$P_n \frac{d^n y}{dx^n} + P_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + a_1 \frac{dy}{dx} + a_0 y = Q(x)$$

To solve this, we find the complementary function $y_0^{(x)}$ which must satisfy the equation when $Q(x) = 0$ and must also contain n arbitrary constants. The method popularly use is by assuming the solution the form

$$y = Pe^{\lambda x}$$

We differentiate n^{th} times thus

$$y = P_0 e^{\lambda x}$$

$$y' = P_1 \lambda e^{\lambda x}, \qquad y'' = P_2 \lambda^2 e^{\lambda x} \text{ up to}$$

$$y^n = P_n \lambda^n e^{\lambda x}, \qquad y^{n-1} = P_{n-1} \lambda^{n-1} e^{\lambda x}$$

and substitute accordingly to obtain the auxiliary equation thus

$$P_n \lambda^n e^{\lambda x} + P_{n-1} \lambda^{n-1} e^{\lambda x} \cdots + P_2 \lambda^2 e^{\lambda x} + P_0 e^{\lambda x} = 0$$

$$P_n \lambda^n + P_{n-1} \lambda^{n-1} \cdots + P_2 \lambda^2 + P_1 \lambda^1 + P_0 e^{\lambda x} = 0$$

This general auxiliary equation has n roots such as $\lambda_1, \lambda_2 \cdots, \lambda_n$

It is important to note from this point that in some cases, some case may be complex. There are three main condition t o be considered here

i. When all roots are real and distinct, the complementary function is of form

$$y_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + c_3 e^{\lambda_3 x} \cdot\cdot\, c_n e^{\lambda_{1n} x}.$$

The solution here is linearly independent of the auxiliary equation

ii. Some roots complex, if one of the roots of the auxiliary equation is complex, such as $\propto + i\beta$, its complex conjugate is also a root. Under this condition,

$$y_c(x) = c_1 e^{(\propto + i\beta)x} + c_2 e^{(\propto - i\beta)x}$$

$$= e^{\propto x}(d_1 \cos Bx + d_2 \sin Bx)$$

$$= Be^{\propto x} \sin(\beta + \phi)x \; or \; \beta\, e^{\propto x} \cos(\beta + \phi)x$$

Where A and ϕ are arbitrary constants.

iii. When some roots are repeated. If some roots are repeated, those set of repeated roots are considered to be linearly dependent, therefore the complementary function is given by

$$y(x) = (c_1 = c_2)x = \cdots c_k x^{(k-1)} e^{\lambda_1 x} + \cdot\cdot\, c_n e^{\lambda_n x}$$

If more roots are repeated it is extended.

Example 11

Solve

$$y'' + y' - 2y = 0$$

Assume $y = e^{\lambda x}$

$$y' = \lambda e^{\lambda x}; \; y'' = \lambda^2 e^{\lambda x}$$

We substitute into the equation

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda + 2)(\lambda - 1) = 0$$

$$\lambda = -2,1$$

$$\lambda_1 = -2, \lambda_2 = 1$$

$$y_c^{(x)} = c_1 e^{-2x} + c_2 e^{x}$$

In this case the roots are distinct.

Example 12

Solve

$$y''' - y'' - 5y' - 3y = 0$$

Solution

The auxiliary equation is

$$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$$

$$(\lambda + 1)(\lambda - 3)(\lambda +) : \lambda_1 = -1, \lambda_2 = 3, \lambda_3 = -1$$

Here we have repeated roots

$$\lambda_1 \text{and } \lambda_3.$$

$$\therefore y_1 = e^{-x}, y_2 = e^{3x} \; and \; y_3 = e^{-x}$$

The function is

$$y_c(x) = c_1 e^{-x} + c_2 e^{-x} + c_3 e^{3x}$$

$$= (c_1 + x c_2) e^{-x} + c_3 e^{x}$$

In this example, the root are repeated since

$\lambda_1 = \lambda_3$.

Example 13

Solve the equation

$$y''' + 5y'' + 9y' + 6y = 0$$

$$\lambda^3 + 5\lambda^2 + 9\lambda + 6 = 0 \text{ is the auxiliary equation}$$

$$\therefore (\lambda + 2)(\lambda^2 + 3\lambda + 3) = 0$$

$$\lambda_1 = -2$$

We now solve

$$\lambda^2 + 3\lambda + 3$$

$$\lambda = \frac{3 \overset{\pm}{} \sqrt{9-12}}{2} = \frac{3}{2} - \frac{\sqrt{3i}}{2}, \frac{3}{2} + \frac{\sqrt{3c}}{2}$$

One real root and two complex root

$$\therefore y_1 = c_1 e^{-2x}, \qquad y_2 = c_2 e^{\left(\frac{3}{2} - \frac{\sqrt{3i}}{2}\right)x}$$

The function is

$$y_3 = c_3 e^{\left(3/2 + \frac{\sqrt{3i}}{2}\right)x}$$

$$y_c(x) = c_1 e^{2-x} + c_2 e^{3/2^x} \sin\sqrt{3/2}\, x + c_3 e^{3/2^x} \cos\left(\sqrt{3/2\, x}\right)$$

$$y = c_1 e^{-2x} + e^{3/2^x}\left(d_1 \sin\sqrt{3/2\, x} + d_2 \cos\sqrt{3/2\, x}\right)$$

Example 14

We solve example 7 using this method. The equation is

$$\frac{d^2x}{dt^2} = -9x$$

$$\lambda^2 = -9$$

$$\lambda = \pm\sqrt{-9}; \Rightarrow \pm 3i$$

$$x = Ge^{3it} + C_2 e^{-3it}$$

$$x = G \sin 3 + + C_2 \cos 3t$$

Is the general solution

However, the elaborate solution will involve the boundary conditions can better be used if we use the method as below.

$$\Rightarrow \frac{d^2x}{dt^2} = V\frac{dv}{dx} = -9x$$

Where we integrate directly

$$\therefore \int vdv = -9\int xdx$$

$$\frac{1}{2}V^2 = -\frac{9}{2}x + C$$

Now it is specified that when $V = 0x = 2$

$$\therefore 0 = \frac{9}{2}(2)^2 + C$$

$$\therefore C = \frac{36}{2}; substituting,$$

$$\frac{1}{2}V^2 = -\frac{36}{2}{}^9/_2\, x^2$$

$$V = \sqrt[3]{4 - x^2}$$

$$dx/dt = \sqrt[3]{4 - x^2}; interm\ of\ \frac{dx}{dt}$$

The change of variable used here is that

$$V = \frac{dx}{dt} \Rightarrow V\frac{dv}{dx} = \frac{d^2x}{dt^2}.$$

$$\therefore \frac{dx}{dt} = \sqrt[3]{4 - x^2}$$

$$\int \frac{dx}{\sqrt{4 - x^2}} = 3\int dt$$

Integration of $\int \frac{dx}{\sqrt{4-x^2}} = \cos^{-1}\frac{x}{2}$

Thus $\cos^{-1}\frac{x}{2} = 3t + \phi$

$$\phi = cos\acute{a}nt$$

$$\therefore \frac{x}{2} = \cos(3t + \phi)$$

$$x = 2\cos(3t + \phi)$$

From this solution, it is obvious that for this type of problem, the method of change of variable gives a better approach to the solution.

For instance, if we consider this problem

$$(1 + x^2)y'' + 2xy' = 0$$

Solution

Let $\frac{dy}{dx} = P: \frac{d^2y}{dx^2} - \frac{dp}{dx}$

$$\Rightarrow (1 + x^2)\frac{dp}{dx} = 2xp$$

$$\frac{dp}{dx} = \frac{-2x}{1 + x^2}P$$

$$\int \frac{dp}{dx} = -\int \frac{2x}{1 + x^2}dx$$

$$\ln P = 1\ln A\,(1 + x^2) \Rightarrow P = \frac{A}{1 + x^2}$$

But recall that $\frac{dy}{dx} = P$

$$\therefore \frac{dy}{dx} = \frac{A}{1 + x^2}$$

$$\int dy = \int \frac{Adx}{1 + x^2} = A\tan^{-1} + B$$

$$\therefore y = A\tan^{-1}x + B$$

Example 15

$$\frac{d^2x}{dt^2} + 4x + 8 = 0$$

$$\frac{d^2x}{dt^2} + 4(x + 2) = 0$$

Let $n = x + 2$

$$\frac{dn}{dt} = \frac{dx}{dt}$$

$$\frac{d^2n}{dt^2} = \frac{d^2x}{dt^2}$$

$$\frac{d^2n}{dt} = -4n$$

Let $\frac{dn}{dt} = P \Rightarrow \frac{d^2n}{dt^2} = P\frac{dp}{dn}$

If we now substitute back in the expression

$$P\frac{dp}{dn} = -4n$$

$$\int pdp = -4\int ndn$$

$$\Rightarrow \frac{1}{1}p^2 = 4\frac{1}{1}n^2 + c$$

When $p = 0,\ n = a$

Such that $c = 2a^2$

$$\frac{p^2}{2} = 2a^2 - 2n^2$$

$$\frac{dn}{dt} = \pm 2\sqrt{a^2 - n^2}$$

$$\int \frac{dn}{\sqrt{a^2 - n^2}} = -\int 2dt$$

$$n = a\cos(2t + \phi)$$

$$x + n = a\cos(2t + \phi)$$

$$B.C$$

When $x = -1, t = 0$

$$x + 2 = a\cos 2t \cos\phi - a\sin 2t \sin\phi$$

$$1 = a\cos\phi, \quad -1 = a\sin\phi$$

$$\Rightarrow \tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{1}{-1}$$

$$\phi = \tan^{-1}(-1) = {}^{\pi}/_{4}$$

Now $a = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$$\therefore x + 2 = \sqrt{2}\cos\left(2t + {}^{\pi}/_{2}\right)$$

$$x = \sqrt{2}\cos\left(2t + {}^{\pi}/_{2}\right)$$

From example 5, if one is asked find the velocity V as a function of time and show that V tends to a terminal velocity as t tends to infinity, find also the time required for such a mass starting at rest to reach 99% percent of its terminal velocity in terms of the acceleration due to gravity and terminal velocity.

Already the differential equation describing this idea has been formed thus

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} - g = 0$$

Which the same as

$$\frac{dv}{dt} + kv = g \; interms \; of \; velocity$$

The auxiliary equation is

$$\lambda + k = 0$$

$$\Rightarrow \lambda = -k$$

$$\therefore v_c = ce^{-kt}$$

We can rewrite the differential equation in term of operator D in order to the particular solution v_p.

$$DV + VK = g$$

$$(D + K)V = g$$

$$v_p = \frac{g}{D + K} = \frac{1}{K}(1 - D)g \cong {}^{g}/_{k}$$

The general solution is now formed by superposing the complementary function and the particular solution v_p.

$$V = v_c + v_p = ce^{-kt} + {}^{g}/_{k}$$

$B.C$, as$t \to \infty$, the terminal velocity is now attained.

$$v_t = {}^{g}/_{k} + ce^{-\infty} = {}^{g}/_{k}$$

$$v_t = {}^{g}/_{k}.$$

Again at $V_{(o)}$,

$$O = {}^{g}/_{k} + ce^{-ko}$$

$$\Rightarrow C = -{}^{g}/_{k}$$

$$V = {}^{g}/_{k} - {}^{g}/_{k}\, e^{-kt}$$

$V = {}^{g}/_{k}\,(1 - e^{-kt})$is the velocity at any given time,

The timer for the mass to reach 99% can be obtained thus

$$\frac{V}{V_t} = \frac{{}^{g}/_{k}\,(1 - e^{-kt})}{{}^{g}/_{k}} = 1 - e^{-kt} = \frac{99}{100}$$

$$\frac{V}{V_t} = 1 - e^{-kt} = \frac{99}{100}$$

$$\Rightarrow 1 - e^{-kt} = \frac{1}{100} \Rightarrow e^{kt} = 100$$

$$kt = \ln 100$$

$$t = \frac{\ln 100}{k}$$

but$V_t = {}^{g}/_{k} \Rightarrow k = {}^{g}/_{V_t}$

$$\therefore t = \frac{\ln 100}{{}^{g}/_{V_t}}$$

$$t = \frac{V_t \ln 100}{g}.$$

When the differential equation is not homogeneous,the solution take stages first obtaining the complementary function by using the characteristics equation and the particular integral associated the equation making the complete solution to be of the form,

$$y = y_C + y_P$$

The last method we have already treated has given as the technicalities of finding y_C

For constant co-efficient, it now remains how to find a particular solution. If a linear differential equation with constant coefficient is given by

$$f(D)y = Q,$$

Then a particular integral is

$$y_P = f(D)Q.$$

The quantity $\frac{1}{f(D)}Q$ is evaluated using any of the methods below as we shall out lone.

i. Recalling that $f(D)$ is a polynomial in D, we can factorize it as a product of n linear expression in D.

i.e

$$f(D) = (D - a_1)(D - a_2) \cdots (D - a_n)$$

In this case, if we consider,

$$D^2 - 2D - 8, = (D - 4)(D + D = (D - 4)(D - [-2])$$

Then $y_p = \frac{1}{f(D)}Q = \left[\frac{1}{D-a_1}, \frac{1}{D-a_2}, \frac{1}{(D-a_3)}\right]Q$

This right-hand side is evaluated step by step taking the first factor the right with Q.

As regards the example given above if Q is given as e^{3x},

$$y_p = \frac{1}{(D-4)}e^{3x}\frac{1}{(D+2)}e^{3x}$$

Let $u = D - 4$

$$(D + 2)U = e^{3x}$$

$$DU + 2U = e^{3x}$$

$$\frac{du}{dx} + 2u = e^{3x}$$

Find the integrating factor, $e^{\int p\,dx} = e^{\int 2\,dx} = e^{2x}$

$$\frac{due^{2k}}{dx} = e^{3x} - e^{2x} = e^{5x}$$

$$\frac{due^{2x}}{dx} = e^{5x}$$

$$ue^{2x} = \int e^{5x}\,dx = \frac{1}{5}e^{5x}$$

$$u = \frac{1}{5}e^{3x}$$

Substituting in the original equation,

$$(D-4)U = \frac{1}{5}e^{3x}$$

$$\frac{du}{dx} - 4u = \frac{1}{5}e^{3x}$$

Using integrating factor

$$e^{\int -4\,dx} = e^{\int -4x}$$

$$\frac{due^{-4x}}{dx} = \frac{1}{5}e^{3x}e^{-4x}$$

$$Ue^{-4x} = \frac{1}{5}\int e^{3x}\,dx\, e^{-4x} = \frac{1}{5}\int e^{-x}\,dx$$

$$Ue^{-4x} = -\frac{1}{5}e^{-x}$$

$$U = -\frac{1}{5}e^{3x}$$

This implies that

$$y_p = -\frac{1}{5}e^{3x}$$

This if one gets y_c which is

$$y_c = C_1e^{4x} + C_2e^{-2x}$$

Then,

$$y = C_1e^{4x} + C_2e^{-2x} - \frac{1}{5}e^{3x}$$

(ii) The second method involves a situation where $\frac{1}{f(D]}$ is expressed as sum of n partial fractions in D.The partial fraction is first resolved.

example,

$$D^2 - 2D - 8e^{3x}$$

$$y_c = C_1e^{4x} + C_2e^{-2x}$$

$$y_p = \frac{1}{(D^2 - 2D - 8)} e^{3x} - \frac{1}{(D-4)(D+2)}$$

First we resolve the denominator in partial fraction.

This gives

$$\frac{1}{6}\left[\frac{1}{D-4} - \frac{1}{D+2}\right] e^{3x}$$

$$\therefore y_p \qquad = \frac{1}{6}\frac{e^{3x}}{D-4} - \frac{1}{6}\frac{e^{3x}}{D+2}$$

$$\frac{1}{6}\frac{e^{3x}}{(3-4)} - \frac{1}{6}\frac{e^{3x}}{3+2}$$

$$= -\frac{e^{3x}}{6} - \frac{e^{3x}}{30} = -\frac{1}{6}e^{3x}$$

$$y = C_1 e^{4x} + C_2 e^{-3x} - \frac{1}{5}e^{3x}$$

Example17

Solve

$$D^3 - D^2 - 8D + (2)y = xe^{2x}$$

Solution

Obtain y_c:

$$D^3 - D^2 - 8D + 12 = (D-2)(D-2)(D+3)$$

$$\Rightarrow y_c = C_1 e^{2x} + xC_2 e^{2x} + C_3 e^{-3x}$$

Now $y_p = \frac{1}{(D-2)(D-2)(D+3)} xe^{2x}$

$$= \frac{1}{D-2}, \frac{1}{D-2}, \frac{1}{D+3}, xe^{3x}$$

We take $D - 2$ one at a time since there are two of them.

Let $U = D + 3$

$$(D-2)U = xe^{3x}$$

$$DU - 2U = xe^{2x}$$

$$du/dx - 2u = xe^{3x}$$

The integrating factor is e^{-2x}

$$\therefore \frac{d}{dx} Ue^{-2x} = xe^{3x}, e^{-2x}$$

$$Ue^{-2x} = \int x\, dx = \frac{1}{2}x^2$$

$$U = \frac{1}{2} x\, e^{2x}$$

Again

$$(D-2)U = \frac{1}{2}x^2\, e^{2x}$$

$$\frac{du}{dx} - 2u = \frac{1}{2}x^2\, e^{2x}$$

Integrating factor $= e^{-2x}$

$$\therefore \frac{du\, e^{-2x}}{dx} = \frac{1}{2}x^3\, e^{2x}e^{-2x}$$

$$Ue^{-2x} = \int \frac{1}{2}x^2\, dx$$

$$Ue^{-2x} = \frac{1}{2}.\frac{1}{3}x^3$$

$$U = \frac{1}{6}x^3 e^{2x}$$

Since $U = D + 3$, we now finally use

$$\frac{dy}{dx} + 3y = \frac{1}{6}x^3 e^{2x}$$

Interacting factor here is e^{3x}

$$ye^{3x} = \frac{1}{6}\int x^3\, e^{5x} dx$$

The right hand side can be solved by using integration by Path

$$\Rightarrow \int x^3\, e^{5x} dx = \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6x}{125}e^{5x} - \frac{6}{625}e^{5x}$$

$$\therefore y_p e^{3x} = \frac{e^{5x}}{5}\left[x^3 - \frac{3}{5}x^2 + \frac{6x}{25} - \frac{6}{125}\right]$$

$$y_p = \frac{e^{5x}}{5}\left[\frac{x^2}{1} - \frac{3x^2}{5} + \frac{6x}{25} - \frac{6}{125}\right].$$

Then the solution of the differential equation is

$$y = y_{c} + y_p$$

$$y = C_1 e^{2x} + xC_2 e^{2x} + C_3 \bar{e}^{3x} + \frac{e^{3x}}{5}\left[x^2 - \frac{3x^2}{5} + \frac{6x}{25} - \frac{6}{125}\right].$$

Example: 18

Solve$(D^4 - 3D^2 - 4)y = 3\cos 3x$

Solution obtain the characteristic equation $C - E$.

$$C.E = m^4 - 3m^2 - 4 = 0$$

$$let\ n = m^2$$

$$\therefore n^2 - 3n - 4 = 0$$

$$(n+1)(n-4) = 0$$

$$n = -1, n = 4$$

$$M^2 = -1; m\ i, -i$$

$$M^2 = 4; m = 2, -2 = 0$$

$$\therefore y_c = c_1 e^{-1x} + c_2 e^{1x} + c_3 e^{2x} + c_4 e^{-2x}$$

$$y_c = c_1 \cos x + c_2 \sin x + c_3 e^{2x} + c_4 e^{-2x}$$

To obtain y_p;

$$y_p = \frac{1}{(D^2)^2 - 3D^2 - 4} 3\cos 3x$$

Now, at each part in time we replace D by the negative argument of $\cos ine$ which is -3

Thus $\frac{1}{(D^2)^2 - 3D^2 - 4} 3\cos 3x\ becomes$

$$\frac{1}{(-3^2)^2 - 3(-3^2) - 4} 3\cos 3x$$

$$\frac{1}{81+27-4}3\cos 3x=\frac{3}{104}3\cos 3x$$

$$\therefore y=c_1\cos x+c_2\sin x+c_3e^{2x}+c_4e^{-2x}+\frac{3}{104}3\cos 3x$$

Example 19

$$D^3(D^2+1)=x^2$$

$$C.E: M=o,o,o,i,-i$$

$$y_1=1, y_2=x, y_3=x^2, y_4=\cos x\, y_5=\sin x$$

$$\therefore y_c=c_1 1 c_2 x+c_3x^2+c_4\cos x+c_5=\sin x$$

Then $y_y=$

$$\frac{1}{D^3(D^2+1)}x^2=\frac{1}{D^3}[D^2+1]^{-1}x^2$$

We expand $[D^2+1]^2$ using Maclaarins series which gives

$$[1-D^2.D^4-D^6+\cdots]x^2$$

$$\Rightarrow\frac{1}{D^3}[1-D^2-D^4-D^6+\cdots]x$$

$$y_p=\left(\frac{1}{D^3}-\frac{1}{D}+D-\frac{1}{D^3}\right)x^2$$

Dmeans differentiate,

D^2differentiate twice

D^3 ⋯ thrice etc.

$\frac{1}{D}$means integrate

$\frac{1}{D^2}$integrate twice etc.

$$\frac{1}{D^3}x^2-\frac{x^2}{D^2}+x^2-D^3x^2$$

$$\Rightarrow\frac{x^5}{60}-\frac{x^3}{3}+2x+0$$

$$\therefore y=y_c+y_p$$

$$= c_1 + xc_2 + c_3x^2 + c_4\cos x + c_5\sin x + \frac{x^5}{60} - \frac{x^3}{3} + 2x.$$

Questions/Exercise 6.0

1. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{4x}$

$$Abs\left[c_1e^x + c_2e^{2x} + {e^{4x}}/{3} + c_4e^{-2x}\right].$$

2. Solve $(D^2 + 9)y = 0$

$$Ans:\ {3}/{5}\,e^{-2x} + c_1\sin 3x + c_2\cos 3x.$$

3. $(D - 3)y = x^3 + 5$

$$Ans:\ -\frac{x^3}{3} - \frac{2x}{9} - \frac{47}{27} = 0$$

4. Solve $(D^2 + 1)(D^2 + 9)y2\sin 2x + \cos 3x.$

$$Ans:\ c_1\cos 3x + c_2\sin x + c_3\cos\sqrt{3x} + c_4\sin\sqrt{3x} - \frac{2}{15}\sin 3x$$

CHAPTER 6

6.10 SERIES OF SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

Series solution is one of the methods used in solution of differential equations. Series solution methods are as outlined below.

6.1.1 The Leibnitz-Maclaurin method

Linear differential equations with variable coefficients can often be solves by assuming a solution in the form of a power series in x. One of the simplest ways of doing this is to use the LEBINTZ-MACLAURIN method as shown

Example

To solve the equation

$$(1-x^2)\frac{d^2y}{dx^2}-5xdy\frac{dy}{dx}-3y=0 \qquad 6.1$$

Solution

We first differential in times using Leibnitz formula

$$(1-x^2)y^{(n+2)}-x(2n+5)y^{(n+1)}-(n+1)(n+3)y^{(n)}=0 \qquad 6.2$$

If we assume as solution of 1 in the form of maclaurin expansion of y then

$$y=y(0)+y^{(1)}(0)+\frac{x^2}{21}y^2(0)+\cdots+\frac{x^r}{ri}y^{(r)}0+ \qquad 6.3$$

Where $y(r)0$ denote the value of dy/dx at $x=0$

The values of these differential coefficients may now be found with the help of the recurrent relation.

$$y^{(n+2)}(0)=(n+1)(n+3)y^{(n)}(0),(nz0)6.4$$

Obtained from (2) by putting $x=0$

Hence we have

$n=0y^2(0)=1.3.y^{(0)}$

$n=1y^3(0)=2.4.y^{(1)}(0)$

$n = 1y^{(4)}(0) = 3.5.\, y^2(0) = 1.3^2.\, y(0)$

$n = 1y^{(5)}(0) = 4.6.\, y^2(0) = 2.4^2.6\, y^{(1)}(0)$

$n = 1y^{(6)}(0) = 5.7.\, y^{(4)}(0) = 1.3^2.5^2.7\, y(0)$etc.

In fact the values of all the differential coefficient at $x = 0$ can, in this way, be expressed in terms of $y(0)$ and $y^{(1)}(0)$

Consequently

$$y = y(0)\left[1 + \frac{1.3x^2}{2!} + \frac{1.3^2 5}{4!}x^4 + 1.3^2.5^2.7x^6 \cdots\right]$$

$$+y^{(1)}(0)\left[x + \frac{2.4}{3!}x^3 + \frac{2.4^2 6}{4!}x^5 + \frac{2.4^2.6^2.7x^6}{7!} + \cdots\right] 6.5$$

Equation (6) may be taken as the general solution of I since it contains two arbitrary constants $y(0)$, which are fixed by specifying boundary conditions on the solution suppose, for e.g. that (1) in to the solved subject to $y(0) = 0$,

$y^{(1)}(0) = 1,$ then (6) Becomes

$$y = x + \frac{2.4}{3!}x^3 + \frac{2.4^2 6}{4!}x^5 + \frac{2.4^2.6^2.8x^7}{7!} + \cdots\right]$$

$$+ \cdots + \frac{2.4^2.6^2}{(2r+1)!}(2r)^2(2r+2)x^{2r+1} + 6.6$$

Which converges, by the ratio test for?$x < 1$

For example ,to obtain solution

$$\frac{x^2 dy}{dx^2} + (1+x)\frac{dy}{dx} + 2y = 0 \; 6.7$$

Near $x = 0$, we first differential the equation n times using Leibnitz's formula. Hence we fine

$$xy_{n+2} + ny_{n+1} + (1+n)y_{n+1} + ny_n + 2y_n \qquad 6.8$$

$$xy_{n+2} + (1 + n + x)y_{n+1} + (n+2)y_n = 0 \qquad 6.9)$$

At $x = o$

$$(1+n)y_{n+1} = -n + 2\,.\,y(0)$$

$$y_{n+1}^{(0)} = -\frac{n+2}{1+n}y_n(0) \qquad 6.10$$

This recurrence relation enables the co-efficient $y(0)$ in true maclaurin expansion

$$y = y^{(0)} + xy_1(0) + \frac{x^2 y(0)}{2!^2} + \cdots \frac{x^r}{r!} y_r(0) + \cdots \qquad 6.11$$

From equation (6.10)

When $n = 0$ $y_1(0) = -2y(0)$

$$n = 1 \quad y_2(0) - {}^3/_2\, y_1(0) = 3y(0)$$

$$n = 2 \quad y_3(0) - {}^4/_3\, y_2(0) = -4y(0)$$

$$n = 23 \quad y_4(0) - {}^5/_4\, y\ {}^2_3(0) = -5y(0) \qquad 6.12a$$

So substituting in equation (6.11)

$$y = y(0)\left[1 - 2x + {}^3/_{21x^2} - {}^4/_{31x^3} + {}^5/_{41}\, x^4 \cdots + (-1)^r \frac{(r+1)}{r!} x^r + \cdots\right] \quad 6.12b$$

Where $y(0)$ in an arbitrary content whose value will in general be determine by specifying a boundary condition on y.

6.1.2 THE FROBENIUS METHOD

The type of equation to be solved by this method is assumed to have in form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + q(x)y = 0 \cdots \qquad 6.13$$

Where $P(x)$ and $q(x)$ are given function of x.

We now wish to obtain solution in the neighbor of $x = 0$. In order to do this $P(x)$ and $q(x)$ most be such that either both $P(x)$ and $q(x)$ are finite in which case $x = 0$ is called an ordinary point of the equation, or both $xP(2)$ and $x^2 q(x)$ remain finite at $x = 0$ in which case $x = 0$ is called a regular point. If $P(x)$ and $q(x)$ do not satisfy either of these conditions the point $x = 0$ is called an irregular singular point and the frobenius method of solution about $x = 0$ is not then applicable, e.g.

$x = 0$is an ordinary point of

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = 0 \cdots \qquad 6.14$$

$x = 0$is a regular singular point of

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{1}{x3}y = 0 \cdots \qquad 6.15$$

$x = 0$is an irregular singular point of

$$\frac{d^2y}{dx^2} + \frac{1}{x^2}\frac{dy}{dx} + xy = 0 \cdots \qquad 6.16$$

The essence of the Frobenius method is to assume a series solution of the type

$$y = \sum_{r=0}^{\infty} a_r x^{m+r} = x^m(a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r \cdots) \qquad 6.17$$

Where $a_0, a_1, a_2 \cdots a_r \cdots$ and in are constants to be determined. This is moregeneral since it may have no integral values.

E.g.

Consider the equation

$$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \cdots \qquad 6.18$$

Assuming a solution of the form

$$= \sum_{r=0}^{\infty} a_r x^{m+r}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} (m+r)(m+r-1)\, a_r x^{m+r-2}$$

Substituting in equation (18)

$$4\sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-2} + 2\sum_{r=0}^{\infty}(m+r)\, a_r x^{m+r-1}$$

$$= \sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$4\sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-1} + 2\sum_{r=0}^{\infty}(m+r)\, a_r x^{m+r-1}$$

$$= \sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

Adding up the first and second terns

$$\sum_{r=0}^{\infty}[4(m+r)(m+r-1)+2(m+r)]a_r x^{m+r-1}+\sum_{r=0}^{\infty}a_r x^{m+r}$$

$$\sum_{r=0}^{\infty}(m+r)(2m+2r-2+1)a_r x^{m+r-1}+2+=\sum_{r=0}^{\infty}a_r x^{m+r}=0$$

$$2\sum_{r=0}^{\infty}(m+r)(2m+2r-1)a_r x^{m+r-1}+2+=\sum_{r=0}^{\infty}a_r x^{m+r}=0 \qquad 6.19$$

When $r=0$

$$2m(2-1)a_0 x^{m+1}+\sum_{r=0}^{\infty}(m+r)(2m+2r-2+1)^{ar+1}x^{m+r-1}$$

$$+\sum_{r=0}^{\infty}a_r x^{m+r} \qquad 6.20$$

And writing $r+1$ for r in the second term of the equation

$$2\sum_{r=0}^{\infty}(m+r)(2m+2r-1)a_r x^{m+r-1}+2+=\sum_{r=0}^{\infty}a_r x^{m+r}$$

$$+\sum_{r=0}^{\infty}a_r x^{m+r} \qquad 6.21$$

Combining the last two terms of (6.21)

$$2m(2m+1)a_r x^{m+1}+\sum_{r=0}^{\infty}(m+r)(2m+2r-2+1)^{ar+1}x^{m+r-1}\,0 \qquad 6.22$$

If (22) is to be solution to equation (18) for all x_1 then the coefficient of all powers of x in (22) is x^{m-1}; consequently

$$2m(2m-1)a_0=0 \qquad 6.23$$

This equation 22 is called the indicial equation in that it determines the values of the index m, in this case (assuming $a_0 \neq 0$)

$$2m(2m-1)=0$$

$$\Rightarrow m=0, {}^{1}/_{2}$$

The appearance of an indicial equation is a standard feature of the frobenius method.

The requirement is that co-efficient of higher power of $x, (1.2x^{m+r} : r = 0, 1, 2 \cdots)$ leads from the 2nd term of (22) to the recurrence relation.

$$2(m+r+1)(2m+2r+1)a_{r+1} + a_r = 0(r = 0,1,2)$$

$$a_{r+1} = \frac{-a_r}{2(m+r+1)(2m+2r+1)}$$

When $m = 0$

$$a_{r+1} = \frac{-a_r}{2(r+1)(2r+1)}$$

$$r = 0 \qquad a_1 = \frac{-a_r}{2.1.1} = \frac{-a_1}{12}$$

$$r = 1 \qquad a_2 = \frac{-a_r}{2.2.3} = \frac{-a_1}{12} = \frac{+a_0}{41}$$

$$r = 2 \qquad a_3 = \frac{-a_r}{2.3.5} = \frac{-a_0}{6!}$$

In general $a_r = \frac{(-1)}{(2r)!}a_0$

Consequently one solution of (6.18) becomes

$$y = x^0\left[a_0 - \frac{a_0}{2!} + \frac{a_0}{4!} - \frac{a_0}{6!} + \cdots + \frac{(-1)}{(2r)!}x + \right]$$

Which is easily recognizable as the series form of?

$$y = a_0 cos\sqrt{x}$$

$a_0$an arbitrary constant

A second solution of (18) may now be obtained by considering the case of $m = {}^1\!/_2$. Denoting the co-efficient *ar*, by *br*, the recurrence relation because $m = {}^1\!/_2$

$$br + 1 = \frac{-b_r}{2(m+r+1)(2m+2r+1)} = -\frac{-b_r}{2({}^1\!/_2 + r + 1)(1 + 2r + 1)}$$

When

$$r = 0 \quad b_1 = \frac{-b_0}{2\,{}^3\!/_2\,2} = \frac{-b}{3!}$$

When $r = 1$

$$b_2 = \frac{-b_0}{2.{}^5/_2.4} = \frac{-b_1}{20} = \frac{b_0}{5!}$$

$$r = 2 \qquad b_3 = \frac{-b_r}{2.{}^9/_2.6} = \frac{-b_1}{42} = \frac{+b_0}{71}$$

Generally; $b_r = \frac{(-1)^r}{(2r+1)!} x^r$

Another solution

$$y = x^{1/2} b_0 \frac{-b_0}{3!} x + \frac{b x^2}{5!} + \frac{b_0 x^3}{7!}$$

$$y = x^{1/2}\, b_0 \frac{-b_0}{3!} x + \frac{b_0 x^2}{5!} + \frac{b_0 x^3}{7!} +$$

$$\frac{(-1)^r}{(3r+1)} x^r$$

$$y = b_0 \sin\sqrt{x}$$

General solution is

$$y = A \sin\sqrt{x} + B \sin\sqrt{x}$$

Example

To solve the equation

$$\frac{d^2y}{dx^2} = xy \ldots (1)$$

$$y'' - xy \qquad\qquad (2)$$

Assume as solution of the form

$$y \;\; = \sum_{r=0}^{\infty} a_r x^{m+r}$$

$$y^1 = \sum_{r=0}^{\infty} (m+r) a_r x^{m+r-1}$$

$$y^{11} = \sum_{r=0}^{\infty} (m+r)(m+r-1) a_r x^{m+r-2}$$

Substituting in (1)

$$\sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r-2}\sum_{r=0}^{\infty}a_r x^{m+r}=0$$

Taking $r=0,1,2$

$$a_0 m(m-1)x^{m-2}+a_1(m+10(m)x^{m-1}+a_2(m+2)(m+2)x^m$$

$$\sum_{r=3}^{\infty}a_r\ (m+r)(m+r-1)x^{m+r-2}-\sum_{r=0}^{\infty}a_r x^{m+r+1}=0-(3)$$

Writing $r+3=\sin(3)$

$$a_0 m(m-1)x^{m-2}+a_1(m+1)(m)mx^{m-1}+a_2(m+2)(m+1)x^m$$

$$\sum_{r=3}^{\infty}(m+r+3)(m+r+2)a_{r+3}-a_r x^{m+r+1}=o \qquad (4)$$

$$\left.\begin{array}{r} a_{0m(m+1)m=0} \\ a_1(m+1)m=0 \\ a_2(m+2)(m+1)=0 \end{array}\right\} \ (5)$$

$$a_r+3=\frac{a_r}{(m+r+3)(m+r+2)}(r,=0,1,2,\cdots)$$

The indicial equation gives $m=0$ &1ie from $a_{0m(m+1)m=0}$

$a_1 is$arbitrary constant $a_2=0$

∴Recurrence relation gives

$$a_{r+3}=\frac{a_r}{(m+r+3)(m+r+2)}$$

$$m=0$$

$$r=0$$

$$a_3=\frac{a_0}{3.2}$$

$$r=1 \quad a_4=\frac{a_1}{4.3}$$

$$r=2 \quad a_5=\frac{a_2}{5.4}=0$$

$$r = 3 \quad a_6 = \frac{a_3}{6.5} = \frac{a_0}{6.5.3.2}$$

$$r = 4 \quad a_7 = \frac{a_4}{7.6} = \frac{a_0}{7.6.4.3}$$

$$r = 5 \quad a_8 = \frac{a_5}{8.7} = 0 \quad e.t.c$$

Hence the solution becomes

$$y = a_0 + a_{1x} + \frac{a_0 x^3}{3.2} + \frac{a_1 x_4}{4.3} + \frac{a_{0x^6}}{6.5.3.2}$$

$$+ \frac{a_0 x^7}{7.6.4.3} \cdots$$

$$= a_0 \left[1 + \frac{x^3}{3.2} + \frac{x^6}{6.5.3.2} + \cdots\right] + a_1 \left[x + \frac{x^4}{4.5} + \frac{x^7}{7.6.4.3} + \cdots\right]$$

$a_0 and\ a_1$ are arbitrary constant another possible case is when $m + 1$ here, $a_1 = 0$ whilst $a_2 = 0$ as bale the recurrence relation gives.

$$a_3 = \frac{a_0}{4.3} a_{\ 4} = \frac{a_0}{5.4} = 0$$

$$a_5 = \frac{a_2}{6.5} = 0 \qquad a_6 = \frac{a_3}{7.6.4.3}$$

$$a_7 = \frac{a_4}{8.7} = 0 \qquad a_8 = \frac{a_5}{9.8} = 0$$

$$a_9 = \frac{a_6}{10.9} = \frac{0}{10.9.8.7.6.4.3}$$

Hence the solution corresponding to m &1 is

$$y = x\left(a_0 + \frac{a_0}{4.3} x^3 + \frac{a}{7.6.4.3} x^6 + \cdots\right)$$

6.1.3 BESSEL'S EQUATION

The equation $\frac{x^2 d^2 y}{dx^2} + \frac{xdy}{dx} + (x^2 - v^2)y = o$ 6.23

Where ν real constant is is call Bessel's equation and their solutions are the Bessel function of order$r$. Applying the Frobnenius method and assuming as serves solution of the type

$$y \quad = \sum_{r=0}^{\infty} a_r x^{m+r} \cdots (2)$$

$$y^1 = \sum_{r=0}^{\infty} (m+r) a_r x^{m+r-1}$$

$$y^{11} = \sum_{r=0}^{\infty} (m+r)(m+r-1) a_r x^{m+r-2}$$

Substituting in equation (1)

$$x^2 \sum_{r=0}^{\infty} a_r\ (m+r)(m+r-1) a_r x^{m+r-2} + x \sum_{r=0}^{\infty} (m+r) a_r x^{m+r+1}$$

$$+(x^2 - v^2) \sum_{r=0}^{\infty} a_r x^{m+r+1} = 0$$

$$\sum_{r=0}^{\infty} a_r\ (m+r)(m+r-1) a_r x^{m+r} + \sum_{r=0}^{\infty} (m+r) a_r x^{m+r}$$

$$+ \sum_{r=0}^{\infty} a_r x^{m+r+2} - v^2 \sum_{r=0}^{\infty} a_r x^{m+r} - (3)$$

$$\sum_{r=0}^{\infty} a_r\ (m+r)^2 (m+r-1) a_r x^{m+r+2} + \sum_{r=0}^{\infty} (m+r) a_r x^{m+r} v^2 \sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

$$\sum_{r=0}^{\infty} a_r\ [(m+r)^2 - v^2] a_r x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

When

$r = 0$ from (3)

$$m(m-1)a_{0,}(m+1)ma_1$$

$$m(m-1)a_0 x^m + (m+1)ma_1 x^{m+1} +$$

$$+ \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$$

Putting $r = r + 2$

$$(m-1)(ma_0x^m + m(m+1)a_1x^{m+1} + \sum_{r=0}^{\infty} a_r\,[(m+r)^2 - v^2]$$

$$a_{r+2}x^{m+r+2} + + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$$

$$(m-1)ma_0 = 0 \cdots i(m+1)na_r = 0$$

$$a_r + 2 = \frac{-a_r}{m+2+r3-v^2}\ (r = 0,1,2)$$

$$a_1 = o\ and\ m = \pm V$$

From the indicial equation, $m = 0,1$

From (6) $m = +v$

$$r = 0$$

$$a_2 = \frac{-a_0}{(v+2)^2 - V^2} = \frac{-a_0}{v^2+v+v-V^2}$$

$$a_2 = \frac{-a_0}{2(2v+2)}$$

For $r = 1$ $\quad a_1 = 0$

$$a_4 = \frac{a_2}{(v+4)^2 - V^2} = \frac{1}{[(V^2+8v+16) - v^2]}\frac{a_0}{2(2v+2)}$$

$$a_4 = \frac{a_0}{2.4(2v+4)(2v+2)}$$

The series solution becomes

$$y = a_0x^v\left\{-1\frac{x^2}{2(2v+2)} - 1\frac{x^4\cdots}{2.4(2v+4)(2v+2+}\right\} \quad 6.27$$

Provided V is not a negative integer similarly, with $m = -v$ we obtain from the recurrence relation.

$$a_2 = \frac{-a_0}{(-v+2)^2 - V^2} = \frac{-a_0}{(\overset{2}{v}-4v+4) - V^2}$$

$$= \frac{a_0}{2(2v+2)}$$

For $r = 1$ $a_1 = 0$

When $r = 2$

$$a_4 = \frac{-a_0}{(-v+4)^2 - V^2} = \frac{-a_0}{v^2 - 8v + 16 - V^2}$$

$$a_4 = \frac{+a_0}{2.4(2v-4)(2v-2)}$$

The solution is

$$y = a_0 x^{-v}\left\{1 + \frac{x^2}{2(2v-2)} - 1\frac{x^4}{2.4(2v-2)(2v-4)} + \cdots\right\} \quad 2.28$$

Provided v_1 is not a positive integer.

In proceeding Bessel function, it is assumed that

$$a_0 = \frac{1}{\sqrt[2]{(v+1)}} \qquad 6.29$$

Where $\Gamma(v+1)$ is the gamma function; now with this form of a_0, we dejoin the Bessel function of the first kind and order, $V_1, j_v(x)$ such that

From (7)

$$y = J_r(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(v+r+1)}\left(\frac{x}{2}\right)^{r+2v} \quad 6.30$$

Similarly the second solution (8) becomes

$$y = J_{-v}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(v+r+1)}\left(\frac{x}{2}\right)^{r+2v} \quad 6.31$$

The general solution of Bessel equation for non-integral v is therefore

$$y = A \quad J_v(x) + B \quad J_{-v}(x) \quad 6.32$$

Where A and B are arbitrary constables if $V = n$ where n is a $+ve$ integer, then

Since

$(n+r+1) = (n+r)!, (9)$ Becomes

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)}\left(\frac{x}{2}\right)^{n+2r} \qquad 6.33$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{-n+2r} \qquad 6.34$$

Now the $1^{st}n$ terms of this series are zero since the – function a_0 factorial function of negative integers is infinite. Hence putting $r = n + p$

In 12 we have

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(-n+r)} \left(\frac{x}{2}\right)^{-n+2r} \qquad 6.35$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^r}{p!\,(n+p)!} \left(\frac{x}{2}\right)^{n+2p} \qquad 6.36$$

$$= (-1)^n J_n(x) \qquad 6.37$$

However $J_n\,(x)$ and $J_{-n}(x)$ are linearly dependent. Hence Fresenius method gives only one solution in these circumstances, therefore (633), cannot be taken as the general solution o the Bessel equation

When $n = v$

Finally we note that from 6.36

$$J_0(x) = \frac{1 - x^2}{(1!)^2 2^2} + \frac{x^5}{(2!)^2 2^4} - \frac{x^2}{(3!)^2 2^6} + \cdots \qquad 6.38$$

When $n = 1$

For $n = 1$

$$J(x) = \frac{x}{2}\frac{1 - x^2}{2^3.1!\,2!} + \frac{x^5}{2^5 2!\,3!} - \frac{x^2}{2^7.3! + !} + \cdots 3.39$$

From which it follows that

$$\frac{dJ_0(x)}{dx} = J_1(x)$$

CHAPTER 7

7.1.0 Partial differential equation

Definition: A partial differential equation (PDE) is an equation relating an unknown function (the dependent variable) of two or more variables with one or more of its variable partial derivatives with respect to those variables. The most commonly occurring independent variables are those describing position and time and so on. Obviously most differential equations of physics involve quantities depending on both space and time which most cases involve partial derivatives and so are partial differential equations(PDE's). They occur as (a) partial differential equation of first degree in the dependent variable and (b) partial differential equation of second order.

7.1.1 Classification of PDE's

Here we focus on second order equations in two variables such as the wave equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial t^2} = f(x,t), \quad \text{(Hyperbolic)}$$

Laplace or poison's equation

$$\text{(PDE's)}\frac{\partial^2 \varphi}{\partial x^2}\frac{\partial^2 \varphi}{\partial y^2} = f(x,y), \text{ (Elliptic)}$$

Or Fourier's heat equation.

$$\frac{\partial^2 \varphi}{\partial x^2} - K\frac{\partial^2 \varphi}{\partial t} = f(x,t), \text{ (Parabolic)}$$

What do the names hyperbolic, elliptic and parabolic implies? This steamed that from a co-ordinate geometrical concept that a quadratic curve.

$$ax^2 + 2bxy + cy^2 + fx + gy + h = 0$$

Represents a hyperbola, an ellipse and parabola depending on whether the descriminant, $ac - b^2$, is less than zero, greater than zero or equal to zero.

Similarly, the equation

$$a(x,y)\frac{\partial^2 \varphi}{\partial x^2} + 2b(x,y)\frac{\partial^2 \varphi}{\partial x \partial y} + c(x,y)\frac{\partial^2 \varphi}{\partial y^2} + (lower\ order) = 0$$

Is said to be hyperbolic, elliptic or parabolic at a point (x,y)if

$$\begin{vmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{vmatrix} = |ac - b^2|_{xy}$$

Is less than, greater than or equal to zero, respectively. This classification helps us understand what sort of initial or boundary data we need to specify the problem.

Three classes of boundary conditions as already spelt out in the discussion of order differential equation are also need in solving partial differential equation.

7.1.2 Method of Solution of PDE's

There are different methods of solving PDE's but in this part but we mention solution from direct partial integration which is only limited to simplest form of partial differential equation which we many not worry about to discuss here as it is simply based on direct integration and then we go on to discuss separation of variable. Another method is by use of integral transform and Green's functions separation of variable which is peculiar to second order partial differential equation.

In this approach, we try to separate the variables. That is to keep the independent variable as separate as possible. For instant, if we seek a solution to PDE of the form $U(x,y,z,t)$ we first of all separate the variable thus

$$\varphi(x,y,z,t) = X(x)\,Y(y)\,Z(z)\,T(t)$$

And then solve them separately as by the virtue of the separation, the PDE is reduced to four separate ordinary differential equations which must be connected through four constant parameters that satisfy an algebraic relation. These constants are called separation constants.

Example

Obtain a general solution for 3D wave equation given as

$$\nabla^2\,\varphi(r) = \frac{1}{C^2}\frac{\partial^2}{\partial t^2}\,\varphi(r)$$

Solution

We solve this equation working in Cartesian co-ordinates

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial\varphi}{\partial Z^2} = \frac{1}{C^2}\frac{\partial^2\varphi}{\partial t^2}$$

The first step is to separate the variables as

$$X(x)\,Y(y)\,Z(z)\,T(t) = \varphi(x,y,z,t)$$

The next is to differentiate twice each term respectively.

$$\frac{d^2X}{dx^2}(Y\,Z\,T)+\frac{d^2Y}{dy^2}(X\,Z\,T)\frac{d^2Z}{dz^2}(X\,Y\,T)$$

$$=\frac{1}{C^2}\frac{d^2T}{dt^2}(X\,Y\,Z)$$

This can simply be written as

$$X''\,Y\,Z\,T+X\,Y''\,Z\,T+X\,Y\,Z''\;T=\frac{1}{C^2}T''\,X\,Y\,Z$$

Now we divide through by X Y Z T term by term

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=\frac{1}{C^2}T''$$

Let $-\mu^2=-[l^2+m^2+n^2]$

We can now write

$$\frac{X''}{X}=-l^2,\qquad \frac{Y''}{Y}=-m^2,\qquad \frac{Z''}{Z}=-n^2\quad and\quad \frac{T''}{T}=\mu^2$$

$$\Rightarrow X''+Xl^2=0,\quad Y''+Ym^2=0,\quad Z''+Zn^2=0\;and\;\;T''-C^2\mu T=0$$

These are now ordinary differential equations which are straight forward and have general solution as given below respectively.

$$X(x)=A\exp i(lx)+B\exp-i(lx)(a)$$

$$C\exp i(my)+D\exp-i(my)(b)$$

$$E\exp i(nz)+F\exp-i(nz)(c)$$

$$G\exp i(c\mu t)+H\exp-i(c\mu t)(e)$$

$A,B,\ldots,H\;are$ arbitrary constants that may be determined with given boundary conditions as mayhave be defined in the problem. If we seek a particular solution, then equations (a), (b) (c) and (e) can be written as

$$X(x)=\exp ilx\,,\qquad\qquad Y(y)=\exp iny$$

$$Z(z)=\exp inz\qquad and\qquad T(t)=\exp -ic\mu t$$

The solution is written as the superposition of the solutions term by termis started in the process of separation i.e.

$$X(x)\quad Y(y)\quad Z(z)\quad T(t)=\varphi(x,y,z,t)$$

$$\therefore \varphi(x,y,z,t) = \exp ilx \;.\exp imy \;.\exp inz \;.\exp -ic\mu t$$

$$= expi[lx + my + nz - c\mu t]$$

Note that $l, m \, and \, n$ are vector components of wave called wave number$k$, $c\mu$ is the angular frequency of the wave

$$\therefore \varphi(x,y,z,t) = \exp i \left[K_x x + K_y y + K_z z - wt\right]$$

Example 2

Solve $K\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial\varphi}{\partial t}$

Using separation of variables considering that $t \to \infty$ for all x

Solution

$$\varphi(x,t) = X(x)\,T(t)$$

$$X^{''}T - X\,T^{'} = 0$$

Dividing through by $U = XT$ and then by K

$$\therefore \frac{X^{''}}{X} = \frac{T^{'}}{KT}$$

If we take a $-\lambda^2$ as a constant, then

$$\frac{X^{''}}{X} =- \lambda^2, \quad \frac{T^{'}}{T} =- \lambda^2$$

$$\therefore \frac{X^{''}}{X} + \lambda^2 = 0; \qquad \frac{T^{'}}{T} + \lambda^2 = 0$$

The general solution becomes

$$X(x) = A\cos\lambda\,x + B\sin\lambda\,x$$

$$T(t) = C\exp-\lambda^2\,kt$$

Thus $\varphi(x,t) = [A\cos\lambda\,x + B\sin\lambda\,x]C\exp-\lambda^2\,kt$ using the boundary condition that

$$\varphi(x,t) = X(0)\,T(t) = 0$$

And

$$\varphi(L,0) = X(L)\,T(t) = \;0$$

For $X(0) = 0 \quad = \; A\cos 0 + B\sin 0$

$$\Rightarrow \quad A = 0$$

At $X(L) = B \sin \lambda L = 0$

$$B \neq 0$$

Therefore $\sin \lambda L = 0$

$$\lambda L = \sin^{-1} 0 = n\pi$$

$$\lambda = \frac{n\pi}{L}; \qquad n = 1,2,3, \ldots$$

For the second parts,

$$T^{'} + k\lambda^2 nT = 0$$

$$T(t) = C_n \exp\text{–}\, \lambda^2\, kt$$

$$\varphi(x,t) = C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda^2 kt}$$

$C_n$is a constant.

This is the solution of the heat equation satisfying the boundary conditions.

Example

Determine a solution $U(x,y)$ of he Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

subject to the following boundary conditions.

$$\varphi = 0 \;\; when\; x = 0, \quad \varphi = 0 \;\; when\; x = \pi$$

$$\varphi \to 0 \;\; when\; y \to \infty, \quad \varphi = x \quad when\; y = 0$$

Solution

Using our normal method;

$$\varphi(x,y) = X(x)\, Y(y);\;\; the\ equation\ becomes$$

$$\frac{X^{''}}{X} = \frac{Y^{''}}{Y}$$

Assuming a constant – P^2

We can now write the proceeding equation as

$$\frac{X''}{X} + \rho^2 = 0 \qquad and$$

$$\frac{Y''}{Y} - \rho^2 = 0$$

The first has solution of

$$X = A\cos\rho\, x + B\sin\rho\, x$$

While the second one has solution of the form

$$Y = C\exp\rho\, y + D\exp -\rho\, y$$

The general solution becomes

$$\varphi(x, y) = [A\cos\rho\, x + B\sin\rho\, x][C\exp\rho\, y + D\exp -\rho\, y]$$

Now we apply the first boundary condition;

$$\varphi = 0, x = 0;$$

which reduces the equation to be of the form

$$\varphi(x, y) = \sin\rho\, x[R\exp\rho\, y + Q\exp -\rho\, y]$$

With secondary boundary condition, $\varphi(\pi, y) = 0$

$$\Rightarrow 0 = \sin\rho\,\pi[R\exp\rho\, y + Q\exp -\rho\, y]$$

Now if we recall that

$$\sin\rho\,\pi = 0, \qquad then$$

$$\rho\pi = n\pi; \quad \rho = n$$

Where $n = 1,2,3, \dots$

$$\therefore \varphi(x, y) = sinnx[R\exp n\, y + Q\exp -n\, y]$$

With 3rd boundary condition $\varphi \to 0\ as\ y \to \infty$

$$0 = \sin nx[R\exp n\, y + Q\exp -n\, y]$$

But since it is obvious that

$$\exp -ny \to 0, \qquad then$$

$$\varphi_n(x, y) = \varphi\sin nx\exp -ny$$

For $n = 1$,

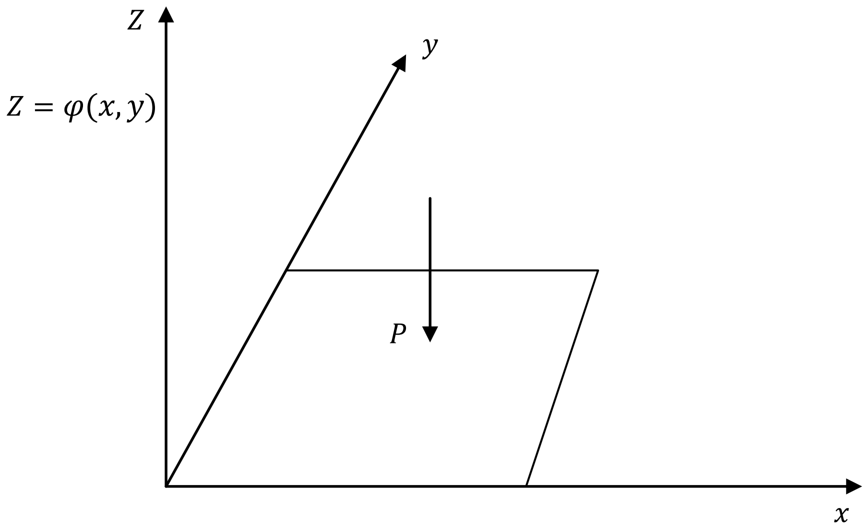
$$\varphi_1(x,y) = \varphi_1 \exp -y \sin x$$

For $n = 2$

$$\varphi_2(x,y) = \varphi_2 \exp -2y \sin 2x$$

$$\varphi(x,y) = \sum_{r=1}^{\infty} Qr \exp -ry \sin rx$$

Becomes the solution at this stage finally with the fourth boundary condition



$$\varphi = 3, \qquad y = 0$$

$$\therefore 3 = \sum_{r=1}^{\infty} Qr \sin rx$$

To obtain the value of Qr, we consider that fact from $0 \; to \; \pi$ equal x is range. Therefore

$$Qr = 2 \times mean\ value\ of\ 3 \sin rx \ \ between\ 0\ to\ \pi$$

$$Qr = \frac{2}{\pi}\int_0^{\pi} 3 \sin rx \ \ dx = \frac{6}{\pi}\left[-\frac{\cos rx}{r}\right]_o^{\pi}$$

$\therefore Qr = 0$ when$r$ is even; and $Qr = \frac{12}{r\pi}$ when r is odd.

$$\therefore \varphi(x,y) = \sum_{r_{edd}=1}^{\infty} \frac{12}{r\pi} \exp -ry \sin rx$$

$$Q(x,y) = \frac{12}{\pi}\left[e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \frac{1}{5}e^{-5y}\sin 5x + \cdots + \frac{12}{r\pi}e^{-ry}\sin rx\right]$$

7.1.3 Laplace's Equation

This type of equation concerns the distribution of fields such as temperature, potential which are scalar fields and other types of filed over a plane area subject to certain boundary conditions.

For example, if we designate the potential at a point P in a plane by an ordinate axis and consider it to be a function of this position as given below,

$$Z = \varphi(x,y)$$

$$where \quad \varphi(x,y)$$

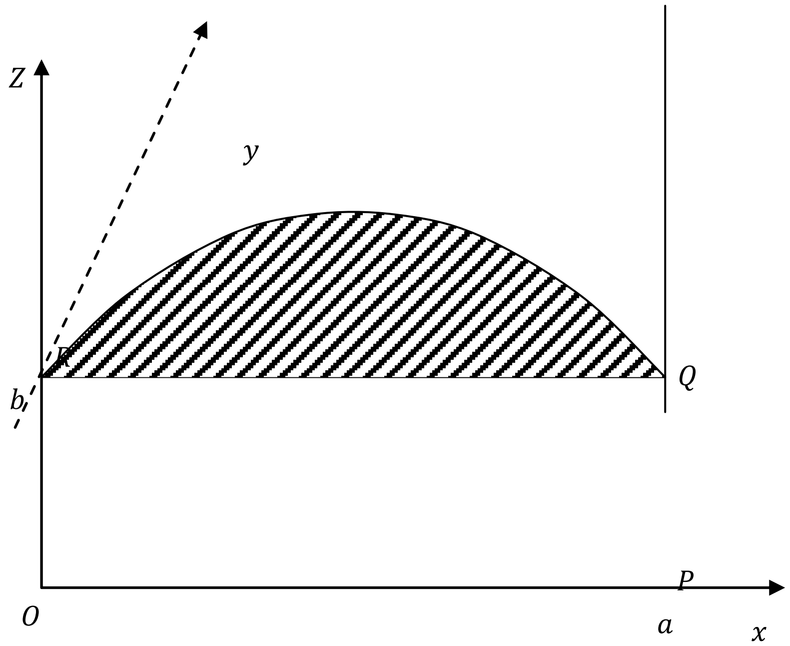
Is a solution of the Laplace two dimensional equation, then

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

At this point, we wish o determine the solution of equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

For a rectangular bounded by line $x = 0, y = 0 \;\; and \;\; x = a \;\; and \; y = b$



With this specification, it implies that

$$\varphi = 0, \quad when\ x = 0, 0 \le y \le b;$$

$$\varphi = 0, \quad when\ y = 0, 0 \le x \le a$$

i.e.

$$\varphi(0, y) = 0\ \ and\ \varphi(a, y) = 0\ \ for, 0 \le x \le a$$

The solution $Z = \varphi(x, y)$ gives the potential at any point within the rectangle $OPRQ$

$$\frac{\partial^2\varphi}{\partial x^2} = X'' Y, \frac{\partial^2\varphi}{\partial y^2} = Y'' X$$

Now,

$$\varphi(x, y) = X(x).Y(y)$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$$

We assume a constant $-P^2$

This enables us to write the proceeding equation as

$$X'' + \rho^2 x = 0\ \ and\ \ Y'' - \rho^2 Y = 0$$

The solution of these equations is

$$X = A\cos\rho x + B\sin\rho x$$

$$Y = C\cosh\rho y + D\sinh\rho y = E\sinh(y + \phi)\rho$$

Now the general solution is

$$\varphi(x,y) = [A\cos\rho x + B\sin\rho x][E\sinh\rho(y+\phi)]$$

$$\Rightarrow [Q\cos\rho x + R\sin\rho x]\sinh\rho(y+\phi)$$

$AE = Q,\ BE = R$are all arbitrary constant. Applying the first boundary condition

$$\varphi(x,y) = 0;\ \ Q\sinh\rho\,(y+\cancel{Q}$$

$$\therefore\ \ Q = 0$$

$$\Rightarrow \varphi(x,y) = R\sin\rho x\sinh 0(y+Q)$$

Applying the secondary boundary condition where we have

$$U(a,y) = 0$$

$$0 = R\sin\rho a\sinh\rho(y+Q)$$

$$\Rightarrow \sin\rho a = 0;\ \ \ \rho a = n\pi$$

Assuming $\lambda = \rho$, then

$$\lambda = \frac{n\pi}{a}$$

$$\therefore \varphi(x,y) = R\sin\lambda x\sinh\lambda(y+Q)$$

Again using the boundary condition (third) which specifies $\varphi(x,y) = 0;\ \ 0 = R\sin\lambda x\sinh\lambda(b+\phi)$

$$\Rightarrow \sinh\lambda(b+\phi)$$

$$\phi = -b$$

$$\therefore \varphi(x,y) = R\sin\lambda\sinh\lambda\,(y-b)$$

$$\varphi(x,y) = \sum_{r=1}^{\infty} R_r\sin\lambda\sinh\lambda\,(b-y)$$

From the fourth boundary condition

$$\varphi(x,0) = f(x)$$

$$f(x) = \sum_{r=1}^{\infty} R_r\sin\lambda\sinh\lambda\,(b-y)$$

$$\therefore R_r\sin\lambda b = 2\ \times mean\ value\ of\ f(x)\sin\lambda x$$

From $x = 0\ to\ x = a$

$$R_r = \frac{2}{a}\int_0^2 f(x)\sin\lambda\, xdx$$

7.1.4 Separation of Variables in Polar Coordinates

In as much as we can express Laplace's equation in cylindrical and spherical polar coordinates such as

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right) \qquad A$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \partial^2/\partial z^2\right) \qquad B$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r^2}\right)\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2} \qquad C$$

These three forms of equation can be solved using separation of variables for instant from equation A we write $\varphi(p,\phi) = p(\rho)\Phi(\phi)$, the equation becomes

$$\frac{\Phi}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{P}{P^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

Divide through by $U = \rho\Phi$ and multiply through by ρ^2, one obtains

$$\frac{\rho}{P}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial P}{\partial\rho} + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0\right.$$

We assume the second term in the equation to be $-n^2$

$$\therefore \frac{1}{\Phi}\frac{\partial^2}{\partial\phi^2}\Phi = -n^2$$

Considering this equation in such a way that$n \neq 0$, the solution becomes

$$\Phi(\phi) = A\exp in\,\phi + B\exp -in\,\phi$$

The original equation becomes

$$\frac{\rho}{P}\frac{\partial P}{\partial\rho} + \frac{\rho^2}{P}\frac{\partial^2 P}{\partial\rho^2} - n^2 = 0$$

$$\rho^2\frac{\partial^2 P}{\partial\rho^2} + \rho\frac{\partial P}{\partial\rho} - n^2 P = 0$$

$$\rho P'' + \rho P' - n^2 P = 0$$

The equation can be solved using power series solution or substitution.

For cylindrical coordinate

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\Phi^2} + \frac{\partial^2\varphi}{\partial z^2} = 0\right.$$

$$\varphi(\rho,\phi,z) = P(\rho)\Phi(\phi)\,Z(z)$$

$$\frac{1}{P\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dZ^2} = 0$$

We consider a separation constant K^2 and associate it to the third of the equation. Thus

$$\frac{1}{P\rho}\frac{d^2Z}{dz^2} = K^2$$

Whose solution is

$$Z(z)\;c\exp -KZ) + D\exp(KZ)$$

The original equation reduces to

$$\frac{1}{P\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2} + K = 0$$

$$\Rightarrow \frac{\rho}{P}\frac{d}{dP}\left(\rho\frac{dP}{d\rho}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + K^2\rho^2$$

After multiplying by ρ^2

We again take another separation constant as m^2 and then associate it to the second term of this proceeding equation thus

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m$$

Whose solution becomes

$$\Phi(\phi) = e\cos m\phi + fD\sin m\phi$$

The original equation becomes now

$$\rho\frac{d}{dP}\left(\rho\frac{dP}{d\rho}\right) + K^2(\rho^2 - m^2)P = 0$$

This final expression can be solved first by transforming it into Bessel's equation which we may get into in this text.

8.1.1 GREEN'S FUNCTIONS

Green's functions play an important role in the solution of linear ordinary and partial differential equations and are a key component to the development of boundary integral equation methods.

Consider a given linear differential equation L(x) U(x) = F(x) …(1)

Where L(x) is a linear, self- ad joint differential operator U(x) is the unknown function with f(x) as a known non0homogeneous term. We can construct the inverse operator $L^{/}(x)$ such that $L^{-1}(x)\ L(x)$ = $L(x)\ L^{-1}(x)$ = 1.

Having obtained $L^{-1}(x)$, then we can now write U(x) = $L^{-1}(x)$ f(x)

L may be a bounded operator or an unbounded operator if L is a differential operator, then it is an unbounded operator.

When L(x) is a differential operator, then the inverse operator L(x) is an integral operator such that $L^{-1}u(x) = \delta x,x^{/} f(x^{/})\ dx^{/}$

Where the kernel $G(x,x^{/})$ is called

Is called a green's function if the last defined equation holds then the green's function $G(x,x^{/}) = \delta(x,x^{/})$ and the solution of the problem can be writer informs of green's function as

$U(x) = G(x,x^{/}) f(x^{/}) dx^{/}$

To prove that this is a solution to the problem,

$Lu(x)u(x) = f(x) = f(x)$

$$LU(x) = L\int_{-0}^{\infty} G(x,x^{/}) f(x^{/})\ dx^{/}$$

We simply substitute as follows $= \int_{-0}^{\infty} LG(x,x^{/}) f(x^{/}) dx$

$$= \int_{-0}^{\infty} \delta(x,x^{/} f(x^{/}) dx^{/} = f(x)$$

This was made possible due to the linearity of the differential and inverse operators.

The green's function can be interpreted physically for variety of differential operator considered in mathematical physics and engineering. It is very interesting and less cumber some. It just involves an approach to the solution of nonhomogeneous boundary-value problem in which the problem is approached first by a means of constructing an auxiliary function known as green's function if we consider two dimensional lap lace's equation

$$\nabla u = \frac{d^2u}{dx_1^2} + \frac{d^2u}{dx_2'^2}$$

Where ∇ is the operator. The green's functions for this particular differential operator is known to be

$$G(x, x') = -\frac{1}{2\pi} ln\ r = -\frac{1}{2\pi}\ \ln \sqrt{(x' - x'_1)^2 + (x^2 - x_2'^2)}$$

We note here that the green's functions gives the potential at the point x due to a point x[1] the source point which this, Green's function only depends on the distance between the source and field point.

Example 22

Consider a stretched string at rest under an external distributed load given by f (x) (force per unit length0. the displacement u of the string is a function of x only and satisfies differential equation

$$T\ \frac{d^2 u\ (x)}{dx^2} = f\ (x)$$

$$with\ u\ (0) = u(L) = 0$$

Solution

Let

Us solve the problem for F_o at the point

Now we seek the solution of the equation

Where we require that $G(0, x') = G\ (L, x') = 0$

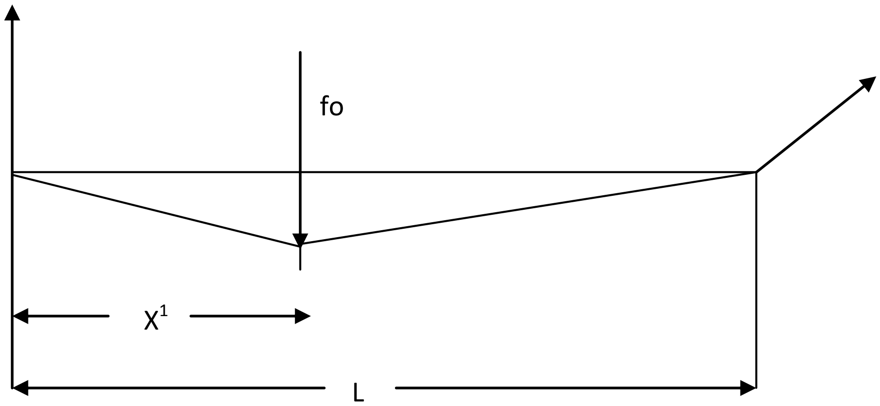
$$this\ G(x, x') = Ax + B \text{ for } 0 \le x \le x'$$

Applying the boundary condition shows that at x=0, b=0 while A remains undetermined

Also as $G(x, x') = A'x + B' \text{for } x' \le x \le L$

With the boundary condition at $x = L$ we obtain $B^1 = -A^1 L$ While A^1 undetermined As $G(x, x')$ appears to be possibly the idealized represented shape of the string, it must be continuous a $x = x'$ which mean that $Ax'(x' - L)$ we now the differential equation as

$$\frac{d^2G}{dx^2} = \sqrt{(x - x')}$$



Between

$x' - \in$ and $x' + \in$ and set $\in \to 0$ to obtain $\frac{dG}{dx}\left[(x'+0), x'\right] - \frac{dG}{dG}(x' - 0, x') = 1$

We now obtain

$\frac{dG}{dx}(x'+0), x'$ *from*

$$G(x, x') = \frac{Ax'}{x'} L(x - L) x \le x' \textit{ where } \text{yields } \frac{dG}{dx\left[(x'+0), x'\right]} = \frac{Ax'}{x - L}$$

$$\textit{similarly} \text{ from } G\left(x_1 x'\right) = Ax\left(x, x'\right)$$

$$\textit{we} \text{ obtain } \frac{dG}{dx}\left(x^1 - 0, x^1\right) = A$$

$$\textit{then,} \text{ from } Ax^{1/\left(x^1 - L\right)} - A = 1 \text{ we obtain}$$

$$A = \left(x' - L\right)/L\left\{x\left(\frac{L - x'}{L}\right)\right\} 0 \le x \le x'$$

$$G\left(x, x'\right) \qquad x'\left(\frac{L - x}{L}\right) x' \le x \le L.$$

We note that the Green's function is symmetric in the variables x and x'

$ie\ G(x,x') = G(x,x')$

Which is a property that is very essential in many applications?

The general solution of the no homogenous equation

$$\frac{d^2u}{dx^2} = \frac{f(x)}{T} \textit{ with } \text{given}$$

Boundary conditions is given by $u(x) = \int_0^L G(x,x') f\left(\frac{x'}{T}\right) dx'$

Example 23

Obtain $\psi(x)$ that satisfied $\frac{d^2\psi}{dx^2} + K^2\psi = -fx$ for $0 \le x \le a$

Subject to the condition $\psi(0) = \psi(a) = 0$

Solution

With green's function method, we must first determine the green's function $G(x,x')$ associated with the d e such G obeys the equation

$$\frac{d^2u}{dx^2} + KG = -\delta(x,x')$$

if G is known then

$$\psi(x) = \int_0^a G(x,x') f\ (x') dx'$$

In constructing green's functions for this problem, we observe some important properties

i The green's function associated with the differential equation satisfies the homogeneous d.e $G'' + k^2G = 0$ in each of the intervals $0 \le x' \angle x$ and $x \le x' \le a$, but not at $x = x'$ because the second derivative of G does not exist at that point.

ii $G(x,x')$ satisfies the boundary condition of the given problem. Thus $G(x,0)=G(x,a)=0$

iii $G(x,x')$ is symmetric as already mentioned ie $G(x,x')=\delta(x,x')$

Now going back to our problem, obtainG of the form

$G(x,x')=A\ \text{Cos Kx}'+B Cos\ \text{Kx}':0\le x'\le x\ and\ G(x,x')=C\ \text{sin Kx}'+D\ Cos\ \text{Kx}' x\le x'\le \text{ a}$

Since in each of the stated interval $G''+k^2G=0$

$G\equiv \text{ A}'\text{ exp ik x}'+D\text{ exp-ikx}$

and the B.c, $\text{G}(\text{x},0)=G(x,a)=0$

$G(x,0)=B=0$ and $\text{G}(\text{x,a})=C$ sin Ka + D Cos Ka = 0

$$D = -\frac{\text{C sin ka}}{\text{Cos k a}}$$

$$\text{A sin kx = C sin kx + D cos kx}$$

$$= \frac{\text{C sin Kx Cos Ka - C sin Kx sin ka}}{\text{cos ka}}$$

$$C = \frac{\text{A sin kx Cos ka}}{\text{A sin Cos Ka -Cos kx sin ka}}$$

$$and\ \text{D} = \frac{\text{C sin ka}}{\text{Cos ka}}$$

Differentiating A sin kx = C sin kx + D cos kx

kC cos kx - kD sin kx =KA cos kx =-,

Substituting for C and D, we obtain

We substitute for the values of C and D respectively in order to enable get A

$$KA\ \text{cos kx sin kx cos ka} + \text{kA sin}^2\text{ kx sin ka -kA cos kx}(\text{sin kx cos ka - cos kx sin ka}) = -(\text{sin kx cos ka + cos kx sin ka})$$

$$\text{Ka cos kx sin kx cos ka + kA sin kx sin ka - kA cos kx sin kx cos ka + kA cos}^2\text{ kx sin ka} =-(\text{sin kx cos ka cos kx sin ka})$$

$$\text{kA sin ka}\left(\sin^2 kx+\cos^2 kx\right)=-\begin{pmatrix}\text{sin kx cos ka -cos kx sin ks}\\ \text{kA sin ka = sin kx cos ka -cos kx sin kx}\end{pmatrix}$$

$$A = \frac{\sin k(x-a)}{k \sin k(a-x)} \text{ and } b=0$$

$$C = \frac{A \sin kx \cos ka}{-\sin k(a-x)} \equiv \frac{-\sin kx \cos ka}{k \sin ka} \text{ and } D \equiv \frac{\sin kx \sin ka}{k \sin ka}$$

$$\therefore G(x,x') = \frac{\sin kx' \sin k(a-x)}{k \sin kx}; \le x' \le x$$

$$= -\frac{\sin kx \cos ka \sin kx' + \sin kx \sin ka \cos x'}{k \sin ka}$$

$$G(x,x') = \frac{\sin kx \sin k(a-x')}{K \sin ka}; x \le x' \le a.$$

This expression is green's function for the differential equation. The final solution of problem can be written as

$$\psi(x) = \int_0^a G(x,x') f(x') dx'$$

$$= \int_0^a \frac{\sin kx \sin k(a-x')}{k \sin ka} \delta(x,x') A dx'$$

8.1.2 SOLUTION INVOLVING SERIES EXPANSION

We can also use series expansion to construct green's function for a

given nonhomogeneousproblem say $y(x) = \int_0^L G(x,x') f(x') dx'$

Green's function $G(x,x_1)$ mast

Therefore satisfy $\frac{d^2G}{dx^2} = \delta(x,x')$

And $G(0,x') = G(L,x') = 0$

Since $G(x,x')$ varnishes at the end of the interval (0,L), it follows that if it the be expanded in a series of chosen orthogonal function such

that, for instance the Fourier sine series $G(x,x') = \sum_{n=1}^{\infty} \gamma_n(x') \sin\left(\frac{n\pi x}{L}\right)$

Where the expansion co-efficient γ_n and depend on x_1

We differentiate twice to obtain $\delta(x,x') = \sum_{n-1}^{\infty} \gamma_n(x') \sin\left(\frac{n\pi x}{L}\right)$

$$\frac{d^2G}{dx^2}(x,x') = \sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{L^2}\right)\gamma_n\ (x')\ \sin\left(\frac{n\pi x}{L}\right)$$

Also if $\Delta_1(x') = \frac{2}{L}\int_0^L \delta(x,x')\sin\frac{n\pi x}{L}dx = \frac{2}{L}\sin\frac{n\pi x'}{L}$

$$\delta(x,x') = \sum_{n-1}^{\infty}\Delta_n(x')\sin\left(\frac{n\pi x}{L}\right)$$

Then

Putting the series into the differential equation for $G(x,x')$ and equating to the coefficient, we obtain

$$\left(-\frac{n^2\pi^2}{L^2}\right)\gamma_n(x') = \frac{2}{L}\sin\left(\frac{n\pi x'}{L}\right)$$

So that

$$G(x,x') = \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x'}{L}\sin\frac{n\pi x}{L}$$

This formula represents exactly the same green's function we had already obtained before. $G(x,x') = x(l-x')0 \le x \le x'$
$-x'(L-x)x' \le x \le L$

Which also can be verified directly by expending it in Fourier since series for the series, we construct the solution as thus

$$y(x) = \int_0^L G(x,x')f(x')\,dx'$$

Which leads to?$(0,L)$,

Where an are the Fourier since coefficient of f(x) green's function many be more valuable for computation if it is known in "closed". For instance, the expression

$$y(x)=\int_0^x x\left(\frac{x-L}{L}\right)f(x')dx'+\int_x^L x'\left(\frac{x-L}{L}\right)f(x')dx'$$

Is usually considered to be simpler than the Fourier series for $y(x)*$

Example 24

A stretched string is subjected to forced vibrations by on external farce $f(x,t)$

per unit length which varies harmonically with time. The equation is given by $T\frac{\partial^2 u}{\partial u^2}-p\frac{\partial^2 u}{\partial t^2}=f(x,t)$ can be represented in the form $f(x,t)=f(x)e^{-\omega t}$

This the solution u (x,t) would have the same time dependence

$u(x,t)=y(x)e^{-\omega t}$

This means that $y(x)$ satisfies the d.e $\frac{d^2y}{dx^2}+k^2y=\frac{f(x)}{T},k^2=\frac{\omega^2}{c^2}$

With the boundary conditions $y(2)=y(L)=0$ we new seek the green's function satisfying $\frac{d^2G}{dx^2}+k^2G=\partial(x,x'),$

$G(0,x')=G(L,x')=0$

Using the method of the Fourier since series (or finite Fourier since transform) ie we multiply both sides of the d.e by sin $\left(\frac{n\pi x}{L}\right)$ and integrate from 0 to 1 $(0,L)$, we obtain

$\lambda_{mn}=-\left(\frac{m^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}\right)(m,n=1,2,3,\ldots)$ is the Fourier coefficient of the function is

$$\therefore \gamma_n(x') = \frac{2}{L} \sin \left(\frac{n\pi \frac{x'}{L}}{k^2 - n^2 \frac{\pi^2}{L^2}} \right) \quad and \; G(x,x') = \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{\left(n\pi \frac{x'}{L} \right)}{k^2 - n^2 \frac{\pi^2}{L^2}} \sin \left(n\pi \frac{x}{L} \right)$$

This result expresses the symmetry nature of green's function and again it shows that the formula fails if $k^2 = \frac{n^2 \pi^2}{L^2}$ for positive integer of n. this prevails when $G(x,x')$ does not exist with $k^2 = \frac{n^2\pi^2}{L^2}$ then the homogeneous equation reads $\frac{d^2y}{dx^2} + \frac{n^2\pi^2}{L^2} y = 0$ which possess nontrivial solution such as sin $\left(n\pi \frac{x}{L} \right)$ which satiates the prescribed boundary conditions.

To summarize the explosion technique of obtaining the green's function, we assume that we want to solve the differential equation $\ell y(x) = f(x)$ where ℓ is a strum –Liouville differential operator. $y(x)$ Must be expected to satisfy the boundary conditions $B\,y(x) = 0$ where B is the boundary condition operator, namely, an expression of the form

$$\beta = \alpha_1 + \alpha_2 \frac{d}{dx} (at \text{ x=a})$$
$$\beta = \beta_1 + \beta_2 \frac{d}{dx} (at \text{ x = b})$$

We now seek for the green's function $G(x,x')$ that will satisfy

$$\ell G = \delta(x,x', \beta G = 0$$
$$\ell u_\lambda(x) = \lambda u_\lambda, \beta u_\lambda = 0$$

If G exists and if the set (u_λ) is complete, then G can easily be represented as $G(x,x') = \sum_\lambda \gamma_\lambda(x') u_\lambda(x)$ Applying the operator, ℓ we have

$$\ell G(x,x') = \sum_\lambda \gamma_\lambda \delta u_\lambda(x) = \sum_\lambda \gamma_\lambda(x) \;\; \lambda u_\lambda(x) = \delta(x,x')$$

(x)

if we multiply both sides by $u'_{\lambda}(x)$ and integrate over x

$$\sum_{\lambda}\gamma_{\lambda}(x')\lambda\int_{a}^{b}u'_{\lambda}(x)u_{\lambda}(x)dx=u'_{\lambda}(x')$$

These set of equations can be used to evaluate $\gamma_{\lambda}(x')$ which is easily solved if the eigenfunction $u_{\lambda}(x)$ are or tonsorial, thus

$$\int_{a}^{b}u'_{\lambda}(x)\ u_{\lambda}(x)\ dx=\delta(\lambda\lambda')$$

This leads us to

$$\frac{u_{\lambda}(x')}{\lambda}=\gamma_{\lambda}\left(X'\right)$$

and

$$G(x,x')=\sum_{\lambda}\frac{u_{\lambda}(x')u_{\lambda}(x)}{\lambda},\ \textit{the}\ \text{so}$$

Called bilinear formula that can enable one to write green's function at once if the eigenvalues and eigen functions of ∂ are known very often, if the problem is of the form $\ell y-\lambda y=f$ where λ is an arbitrary parameter the actual eigenvalues of $\sqrt{\ }$ are usually denoted by λ and the bilinear

Formula reads $G(x,x')=\sum_{n}^{0}=\frac{u_{n}(x')u_{\lambda}(x)}{\lambda_{n}-\lambda}$

In complex spaces of functions the bilinear formula is modified to read

$$G(x,x')=\sum_{\lambda}\frac{u_{\lambda}^{*}(x')u_{\lambda}(x)}{\lambda}$$

And Green's function is not symmetric, but rather hermitian under the inter change of x and x/

$G(x,x')=G^{*}(x,x')$As shown by Butkov in his text. 9

8.1.3 FREE SPACE AND REGION DEPENDENT

Green's functions

In the discussion above concerning the solution of a differential equation with a green's function, mention has been made of boundary conditions for the problem because we had not opted to seek for a particular solution. The particular solution is of course independent of any boundary conditions for the problem however, we can always add homogeneous solution to the Green's function.

To obtain a particular

Solution we must always consider the independent of any boundary conditions for the problem but we $L(x)G(x,x_1)=\delta(x,x_1)$ can however add homogeneous solution to Green's function,

$G(x,x_1)=G_0(x,x_.)+G_R(x,x_1)$

where $\mathrm{L}(\mathrm{x})G_R\ (x,x_1)=\delta(x,x_1)$

Go, the particular solution is termed the free space green's function and also referred to as the fundamental solution for the differential operator $L(x)$ and is singular. The homogeneous solution GR is non-singular. Since GR is a homogeneous solution, it will contain constants which can be evaluated to satisfy any boundary conditions, for the problem while the full Green's function $G(x,x')$ is termed region-dependent Green's function since in general, it contains not only particular terms to satisfy any boundary conditions for the problem. Boundary element Methods can be viewed as a symmetric way of constructing numerical approximations to a region dependent, or exact, Green's functions.

Taking a look into Green's function for a partial differential equation we just consider the Helmholtz equation in three dimensions

$$\left(\Delta+k^2\right)u=0$$

$$where\ \Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}$$

Where Δ is the Laplacianoperator. In this case, we seek the Green's function $L(x)G(x,x_1)=-\delta(x,x')$

Where we have to note that three dimensional Dirac delta function is simply product of delta function in each contained

$\delta(x,x_1=\delta(x,x_1')\delta(x_2,x_2'\delta(x_3,x_3'$

To obtain the free-space Green's function for the problem, we apply a Fourier transform method since we will a only be calculating the free-space component of the Green's function one can simply use a single variable $\gamma = x - x'$ as the free-space Green's function will only depend on the relative distance between the source and the filed points and not their absolute position. The flourier transform paid we will use is

$$\overset{\Lambda}{\bigcup}(q) = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty} u(r)\ exp - iqrdr$$

$$= \left(\frac{1}{2\pi^3}\right) \int_{-\infty}^{\infty} \overset{\Lambda}{\bigcup}(q) \exp iqrdq$$

We now apply the forward transform to the differential equation for the Green's function and obtain.

$$\left(q_1{}^2 + q_2{}^2 + q_3 - k^2\right) G(q) = \left(\frac{1}{2\pi}\right)^3$$

Now if we let $q^2 = q_1{}^2 + q_2{}^2 + q_3{}^2$

Then $\left(q^2 - k^2\right) G(q) = \dfrac{1}{(2\pi)^3}$

The Green's function in the transform space is then written as

$$G(r) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\exp(iqr)}{q^2 - k^2} dq$$

The internal is an isotropic Fourier integral since if depends only does on the magnitude of q, which is q, but does depend on the direction of q. Barton in his book elements of green's function and propagation, Oxford science press 1989" gives the general result for isotropic

Fourier integrals in three $\int_{\infty}^{\infty} f(q) \exp iqrdq = \frac{4\pi}{R} \int_{0}^{\infty} qf(R) \sin(qR) dq$

Where R is the magnitude of r, using this result, the inversion integral

we then seek for $G(r) = \frac{4\pi}{(2\pi)3} \frac{1}{R} \int_0^\infty \frac{q}{\left(q^2 - k^2\right)} \sin(\mathrm{qR})dq$

This integral can be evaluated by contour integration, but we first write the sin term in complex exponential.

i.e $\sin(\mathrm{qR}) = \frac{\exp iqR - \exp - iqR}{2i}$

The integral is then written as

$$G(r) = \frac{4\pi}{(2\pi)^3} \frac{1}{4iR} \left\{ \int_{-\infty}^{\infty} \frac{q \exp iqR}{(q-k)(q+k)} dq - \int_{-\infty}^{\infty} \frac{q \exp(-iqR)}{(q-k)(q+k)} dq \right\}$$

$$= \frac{4\pi}{(2\pi)^3} \frac{1}{4iR} \left\{ I_1 - I_2 \right\}$$

I_1 which is the first part of the integral can be evaluated by

considering a contour in the complex q plane. Since the denominator of the integrand has poles on the real axis, imaginary part was introduce in order to effect the poles from the real q axis

$$I_1 = 2\pi i \sum_{\substack{\mathrm{Re}\, s>0 \\ \mathrm{Im}\, q>0}} \frac{qe^{iqk}}{q + (k + i\varepsilon)[q - (k + i\varepsilon)]}$$

$$= \pi i \exp i(k + i\varepsilon)R$$

Obtained by taking a contour in the upper half-plane due to the behaviour of the numerator of the integrand of q becomes large. Using the theory of if the limit as $\varepsilon => 0$ is considered, we have

$$I_1 = -\pi i \exp ikR$$

Similarly using the same process for I_2, we take a contour in the lower half-half to obtain

$$I_2 = -\pi i \exp ikR$$

Now substituting back the value of I_1 andI_2, we take a contour in the lower half-half to obtain

$$G(r) = \frac{4\pi}{(2\pi)^2 \frac{1}{4iR}\{\pi i \exp ikp + \pi\ ienxp\ ikR\}}$$

$$= \frac{1}{4\pi R}\ \exp\ ikR$$

Let us consider a no homogeneous PDE in two space variables such as, for instance, the Poisson equation $\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = f(x, y)$

Specifically, we need to discuss, the state deflection of a rectangular membrane in which the known function $f(x, y)$ represents, the external load per unit area, divided by T (tension) one primary expectation of the deflection $u(a, y), u(x, o) = u(x, b) = 0$ the force F acting at the point (x', n) may be simulated a two dimension function $\frac{F}{T}(\delta x, x') = \delta(y, n)$

Therefore if we solve the equation $\frac{d^2 G}{dx^2} + \frac{d^2 G}{dy^2} = \delta(x, x^2)\delta(y, n)$

In accordance with the superposition principle, we obtain the two dimension Green's function and can possibly represent the solution of the original $P\partial E$ by the integral $u(x, t) = \int_0^a \int_0^b G(x, x'; y, n) dx' dn$

However we use expansion in terms of the eigenfunction $u\lambda(x, y)$ of the lap lace differential operator which must satisfy

$$\nabla^2 \phi\lambda(x, y) = \lambda\phi\lambda(x, y)$$

And the same boundary conditions we are dealing with. The eigenvalues are of the form. $\lambda_{mn} = -\left(\frac{m^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)(m, n = 1, 2, 3, \ldots)$

And the corresponding eigenfunctions read

$$amn = \int_0^a \int_0^b f(x, y)\phi mn(x, y)\, dx\ dy$$

Normalized to unity

We now seek for $G(x, x'; y, n)$ *in* the form

$$G(x, x'; y, n) = \frac{2}{\sqrt{ab}} \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} Amn(x, 1'') \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Substituting the into the PDE for G, we obtain, by standard techniques, $-\left(\frac{m^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right) Amn = \frac{2}{\sqrt{ab}} \sin \frac{m\pi x'}{a} \sin \frac{n\pi}{b} n$

From where obtain

$$Amn = \frac{\frac{-2}{\sqrt{ab}} \sin \frac{m\pi x'}{a} \sin \left(\frac{n\pi}{b}\right) n}{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}$$

If we now substitute for the value of Amn, we obtain immediately

$$G(x, x'; y.n) == \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{\frac{m\pi x'}{a} \sin \frac{n\pi y}{b} \sin \frac{n\pi n}{b}}{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}$$

$$u(x, t) = \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \frac{amn}{m^2 \frac{\pi^2}{a^2} + n^2 \frac{\pi}{b^2}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Where A_{mn} is the expansion co-efficient for $f(x, y)$, given as

$$amn = \int_0^a \int_0^b f(x, y) \phi mn(x, y) dx \, dy$$

8.1.4 Fundamental of Numerical Analysis

Numerical analysis is concerned with the construction of effective methods for the calculation of unknowns entering in the formulation of a given problem. The use of this technique practically involves assumptions and approximations and as such are does not expect high precision than is warranted by the data and the problem.

In recent time the growth of numerical analysis has been enhanced and accelerated by the demands of science and technology problems. As a result of this need, this chapter present the fundamental rudiments of numerical analysis essentially concerned with the processing of numerical data. As for this book we limit ourselves with the basic understanding or principles followed with the acquisition of computing skill. Therefore, emphasis in the following sections is placed on basic ideas and general methods rather than on special techniques in solving this of that problem. Among topics in this chapter are the determination of real roots of algebraic and transcendental equations, the elements of interpolation theory, and its bearing on curve fitting, the numerical equations.

8.1.5 Graphical Method

Geometric considerations usually are a useful guide in the construction of analytic method of solution of practical problems. One can use of Macluarin's and Taylor's theorem in estimation of a given value of a function.

Algebraic equation is a polynomial equation of the type $x^n + a_1 x^{n-1} + \cdots + a_+ = 0$.A transcendental equations are those equation that cannot be reduced to an algebraic equation e.g $\tan x - x = 0, e^x + 2\cos x = 0$.

Example: 1

Use grapy to obtained the real root of the equation

$$f(x) = x^3 - 146.25x - 682.5 = 0$$

We graph the function thus

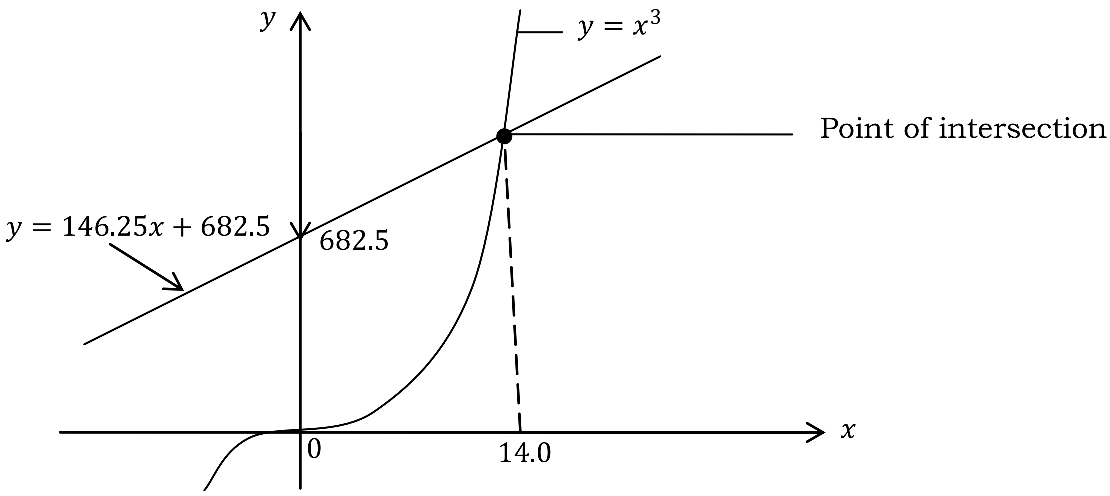
$$y = x^3 - 146.25x - 682.5$$

First we plot the cub

$y = x^3$and the straight line

$$y = 146.25x + 682.5$$

And read off from the graph of abscissa of their point of intersection



8.1.6 Simple Iterative Methods

This method is a better one than the graphical technique. First we have to isolate the real root and apply some iterative formula with an imposition of some restrictions depending on the nature of the function. The simplest of these methods of linear interpolation is known as the method of false position.

Let the root x_0 of $f(x) = 0$ be isolated between a and b. Then in the interval (a, b), the graph of $y = f(x)$ may have the appearance as shown in the graph. If the points A and B are joined by a straight line, it will cut $x - axis$ at w which is a closer root to x_0 than $a$and b. Using similar triangles,

$$\frac{b^1 - a}{-f(a)} = \frac{b - b^1}{f(b)}$$

Solving for b^1,

$$b^1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

This can also be written as

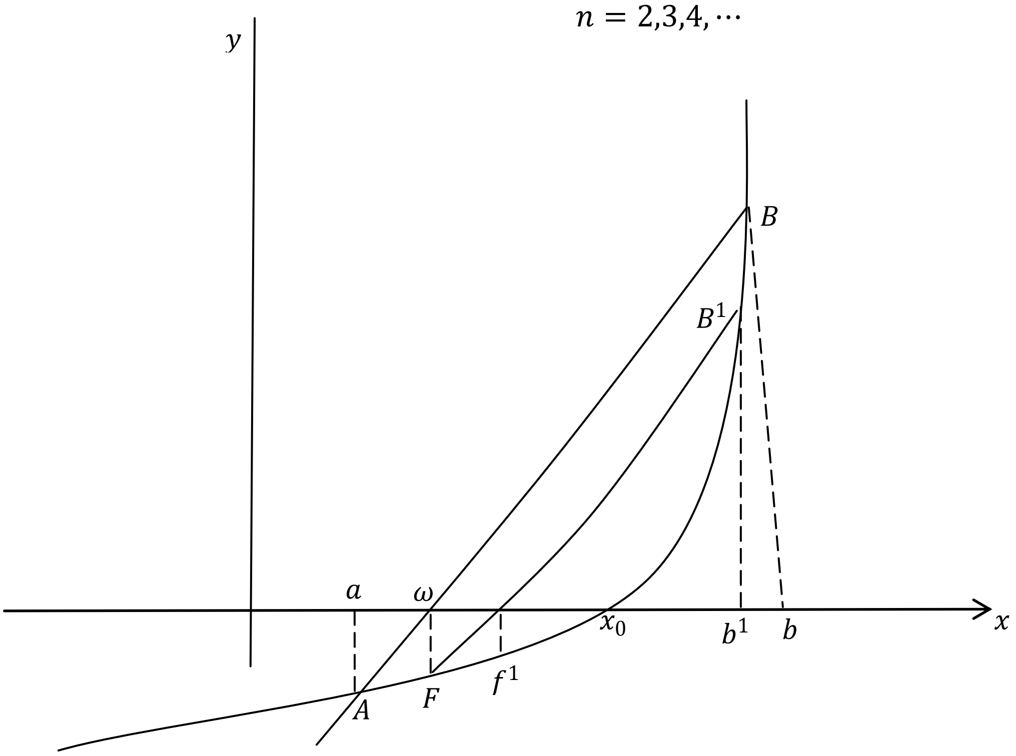
$$b^1 = b - f(b)\frac{b - a}{f(b) - fa}$$

$$\therefore b - b^1 = f(b)\frac{b - a}{f(b) - f(a)}$$

In general to determine the succeeding approximations from the recursion formula we write it in terms of x we have

$$x_{n+1} = x_n - f(x_n)\frac{(x_n - x_1)}{f(x_n) - f(x_1)}$$

$$n = 2,3,4,\cdots$$

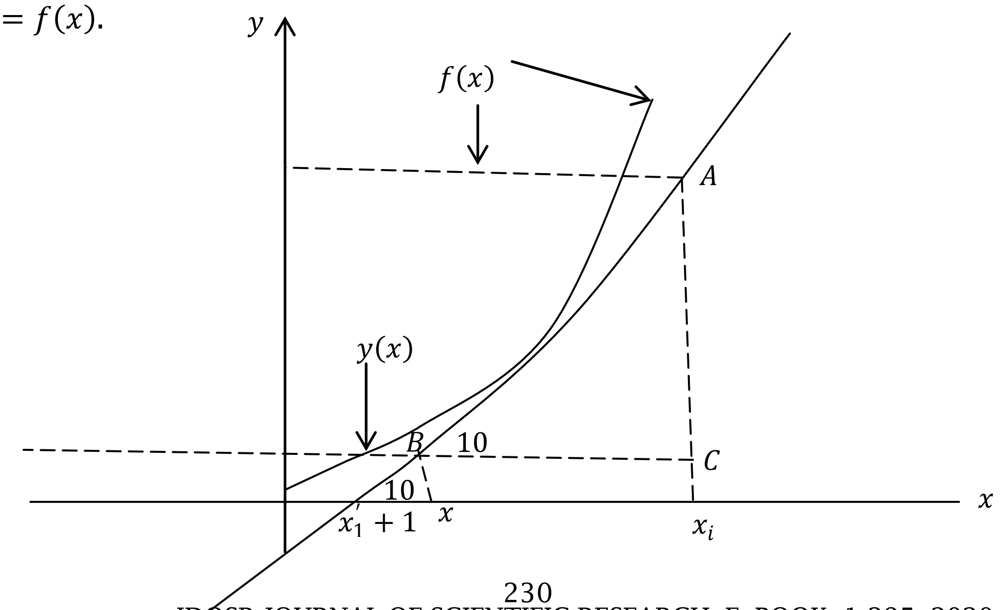


This method is known as method of secants.

8.1.7 Newton's Interaction Method

The successive terms in the approximating sequence in the method of false position involves determining the intersection of the secant line with $x - axis$.

Newton purposed constructing an approximating sequence determine by the intersection with the $x - axis$ of the tangenth line to the curve of $y = f(x)$.



$$Tan\ \theta = \frac{A-C}{C-B} = \frac{dy}{dx} = f^{'}(x_i).$$

Substituting,

$$\frac{f(x_i) - y(x_i)}{x_i - x} = f^{'}(x_i)$$

$$f(x) - y(x) = f^{'}(x)[x_i - x]$$

as$y(x)$ tends to zero; $x = x_{i+1}$

$$\therefore f(x) = f^{'}(x)[x_i - x_{i+1}]$$

$$\Rightarrow x_i - x_{i+1} {f(x_i)}/{f^{'}(x_i)}, x_{i+1} - x_i - \frac{f(x_i)}{f^{'}(x_i)}$$

The geometric considerations is that $y = f(x)$ is a monotone increasing or decreasing function in the interval (x, x_1) so that $f^{'}(x)$ does not change sign and $f(x_1)f^{''}(x) > 0$, the sequence x_{i+1} appears them to converge to the root x_0

Example 2:

Find the root of $x^5 = 60$ using Newton's method.

Solution

$$f(x) = x^5 - 60; \quad f^{'}(x) = 5x^4$$

x_i	$f(x)$	$f^{'}(x)$	${f(x)}/{f^{'}(x)}$	x_{i+1}
1.8	-4.1043	94.4784	-0.4351	2.22506
2.22506	-5.5608	122.557	-0.04456	2.26962
2.26962	0.2231	132.673	0.00682	2.26794
2.26794	6.967×0^{-4}	132.281	5.267×10^6	2.26793

$Ans =$ 2.26793.

Trial Question

Use Newton's method to $3\,d.p$ the real roots of the following equation

(a) $x^4 - x - 1 = 0$ (b) $x^5 - x - 0.2 = 0$

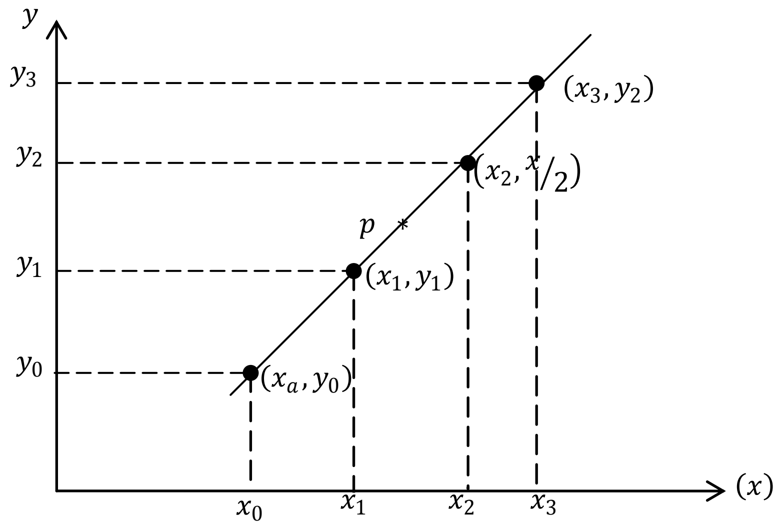
8.1.8 Interpolation

Interpolation is a process of estimating the value of a function for any intermediate value of the variable with the help of certain given values of the function corresponding to a number of variable values.

In other words, if a function $y = f(x)$ is known for values of $x_1, x_2 x_3, \dots, x_n$ as $f(x_1), f(x_2), \dots f(x_n)$, then the process of finding the values $f(x)$ for some other values of x lying between the values $x_1, x_2, x_3, \dots, x_n$. When the estimate of $f(x)$ for some such variable of the function which lies outside the given values, is carried with the help of certain given values of the function corresponding the a number of variable values.

Process is known as extrapolation Langrange's interpolation formulas is one of the interpolation technique developed for use only when the given set of x_i is an arithmetic progression otherwise other type of interpolation formula may be applied.

This method can be derived simply using the diagram below.



Considering any point say p between x_0 and x_1 and y_0 and y_1, we obtain

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

Making y the subject of formula, we have $y = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$

$$\Rightarrow y = a_0 + a_1 x.$$

which can be written as

$$y = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

which we can write as

$$y = \sum_{k=0}^{i} y_k \, b_k^{(x)}$$

$$\exists \qquad b_0 = \frac{x - x_1}{x_0 - x_1}, \qquad b_1 = \frac{x - x_0}{x_1 - x_0}$$

Generally of n^{th} degree of polynomial;

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_0 x^n$$

The points that pas through them can be obtained using the general expression

$$p_n(x) = \sum_{k=0}^{n} y_k \frac{n}{\Pi} \left(\frac{x - x_i}{x_k - x_i} \right)$$

Using equation…, we can

Generate the Langrange formula for the points; $(x_0, y_0), (x_1, y_1), (x_2, x_2), (x_3, x_3)$

$$p_n(x) = y - y_0 \frac{(x - x_1)(x - x_2)}{(x - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$= y_2 \frac{(x - x_0)(x - x_3)}{(x_1 - x_0)(x_2 - x_3)} + \cdots\cdots$$

Example 3

Find the cube root of 5.

Solution:

We assume $x^3 = 5$; $f(x) = x^3 - 5$ let $x_i = 1.5$:; $f'(x_i) = 3x^2$.

i	x_i	$f(x_i)$	$f'(x_i)$	$f(x_i)/f'(x_i)$	x_{i+1}	f/x_{i+1}
1	1.5	−1.625	6.75	−0.2407	1.7407	0.2744
2	1.7407	0.2744	9.0901	0.0307	1.7105	0.0046
3	1.7105	0.00458	8.7774	0.00524	1.0998	0.0000356

Example 4:

Find the value of y when $x = 3$ and 6 respectively using the table

Solution

x	2	4	7
y	12	16	7

In this case, $x = 3, \propto x = 6;$

$$y = 12\frac{(3-4)(3-7)}{(2-4)(2-7)} + 16\frac{(3-2)(3-7)}{(4-2)(4-7)} + 7\frac{(3-2)(3-4)}{(7-2)(7-47)}$$

$$y = 15$$

Repeating with $x = 6$,

$$y = \underline{12}$$

Example 5:

Using the data

v	10	15	22. 5	33.75	50.625	75.937
p	0.300	0.675	1.519	3.717	7.689	17.300

Apply Langrange's formula to find the value of p corresponding to $v = 21$ correct to three $d.p$.

Solution,

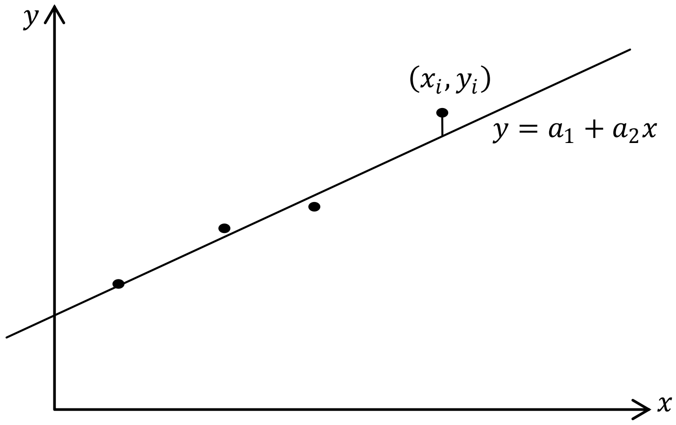
We use 3 nearest values.

$$p = 030\frac{(21-15)(21-22.5)}{(10-15)(10-22.5)} + 0.675\frac{(21-10)(21-22.5)}{(15-10)(22.5)}$$

$$4.1.5\frac{(21-10)(21-15)}{(22.5-10)(22-5-15)} = 1.323$$

8.1.9 Method of Least Squares

This is a method that plotted points by a straight line. $y = a_1 + a_2 k$and choose the parameters a_1 and a_2 so that the sum of the square of the vertical derivations of the plotted points, from this line is as small as possible



If for example we choose to represent set of data (x_i, y_i), write $i = 1,2,...$, by some relationship $y = f(x)$, containing r unknown parameters $a_1, a_2, a_3, ..., a_1$ and form the deviation

$$v_i = f(x_i) - y_i,$$

The sum of the sequence of the deviations

$$s = \sum_{i=1}^{n} v_1^2 = \sum_{i=1}^{n} [f(x_i) - y_i]^2$$

is a function of $a_1, a_2, ..., a_r$. We can then determine the as that s is a minimum. Now, if $s(a_1, a_2, ..., a_r)$ is a minimum, then at the point in equation.

$$\frac{\partial s}{\partial a_1} = 0, \frac{\partial s}{\partial a_2} = 0 \cdots, \frac{\partial s}{\partial a_r} = 0.$$

The set of r equations the above equations called normal equations, serves to determine the r unknown $a's$ in $y = f(x)$.This particular criterion at the “best fit” of data is known as the principle of least squares, and the method of determining the unknown parameters with its and is called ht method least squares. This method was introduced by Gauss.

With this introduction, we elucidate the application of this method by using an example

Example 6:

Calculate the coefficients in $y = a_1 + a_2 x$ to fit the following data;

x	1	2	3	4
y	1.7	1.8	2.3	3.2

Solution:

In this case, $n = 4$

$$\sum_{i=1}^{4} x_i = 1 + 2 + 3 + 4 = 10$$

$$\sum_{i=1}^{4} x_1^2 = 1 + 4 + 9 + 16 = 30$$

$$\sum_{i=1}^{4} y_i = 1.7 + 1.8 + 2.3 + 3.2 = 9$$

$$\sum_{i=1}^{4} x_i\, y_i = 1.7 + 2(1.8) + 3(2.3) + 4(3.2) = 25$$

Using

$$a_1 + \left(\sum_{i=1}^{n} x_i\right) a_2 = \sum_{i=1}^{n} y_i$$

$$\left(\sum_{i=1}^{n} x_i\right) a_1 + \left(\sum_{i=1}^{n} x_1^2\right) a_2 = \sum_{i=1}^{n} x_i\, y_i$$

We substitute accordingly and obtain

$$4a_1 + 10a_2 \neq 9. \qquad (1)$$

$$10a_1 + 30a_2 = 25. \qquad (2)$$

We solve equation 1 and 2 to $a_1$get a_1 and a_2:

$$a_1 = 1, a_2 = \frac{1}{2}:$$

The desired straight line is

$$y = 1 + \frac{1}{2}x$$

which indicates that the graph is fitted into $y = a_1 + a_2x$.

Now, suppose we want to fit by

$$y = a_1 + a_2x + a_3x^3,$$

Then,

$$v_i = a_1 + a_2x_i + a_3x_1^3 - y_i \text{and}$$

$$\frac{\partial v_i}{\partial a_1} = 1, \frac{\partial v_i}{\partial a_2} = x_i, \frac{\partial v_i}{\partial a_3} = x_1^2.$$

The normal equations

$$\sum_{i=1}^{4} v_i \frac{\partial v_i}{\partial a_k} = 0$$

are

$$\sum_{i=1}^{4}(a_1 + a_2x_i + a_3x_i^2 - y_i).1 = 0$$

and

$$\sum_{i=1}^{4}(a_1 + a_2x_i + a_3x_i^2 - y_i)\, x_i^2 = 0$$

Collecting the coefficient of a_i, the normal equations when put in the correct form gives three equations.

$$4a_1 + \left(\sum_{i=1}^{4} x_i\right)a_2 + \left(\sum_{i=1}^{4} x_i^2\right)a_3 = \sum_{i=1}^{4} y_i$$

$$\left(\sum_{i=1}^{4} x_i\right)a_1 + \left(\sum_{i=1}^{4} x_i^2\right)a_2 + \left(\sum_{i=1}^{4} x_i^3\right)a_3 = \sum_{i=1}^{4} x_i\, y_i$$

$$\left(\sum_{i=1}^{4} x_i^2\right) a_1 + \left(\sum_{i=1}^{4} x_i^3\right) a_2 + \left(\sum_{i=1}^{4} x_i^4\right) a_3 = \sum_{i=1}^{4} x_i^2\, y_i$$

The equations become

$$4a_1 + 10a_2 + 30a_3 = 9$$

$$10a_1 + 30a_2 + 100a_3 = 25$$

$$30a_1 + 100a_2 + 354a_3 = 80.8$$

$$a_1 = 2, a_2 = -0.5, a_3 = 0.2$$

The desired straight-line is

$$y = 2 - 05x + 0.2x^2.$$

Trial Questions

Apply the method of least squares to fit the data

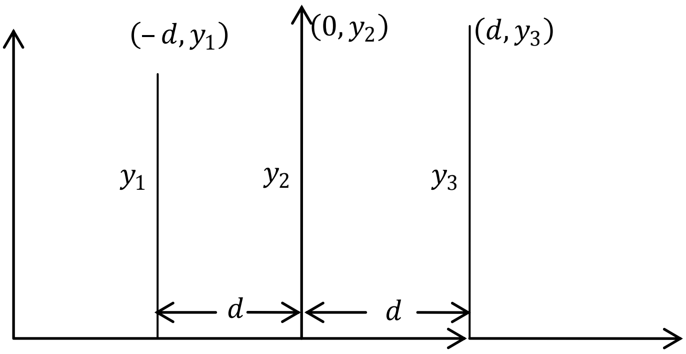
x	1	2	3	4	5	6	7	8
y	2.105	2.808	3.614	4.604	5.857	7.451	9.467	11.985

8.2.0 Numerical Integration

This concept requires a good knowledge of integration and interpretation of the definite integral $\int_b^a f(x)\,dx$ as the area under the curve $y = f(x)$ between the ordinates $x = a$ and $x = b$.This interpretation underlies the construction of formulae for numerical integration as contained in this section. The two aspect of numerical integration as considered this section are Simpson's rule trapezoidal rule.

8.2.1 Simpson's Rule

We use first principle to derive simple expression for the simple's rule let $y = ax^2 + bx + c$ pass through 3 point $(-d, y_1), (0, y_2), (d, y_3)$



Since the curve is parabolic in nature and the parabola passes through 3 pts

$$y_1 = ad^2 - bd + c.. \quad (1)$$

$$y_2 = \quad c.. \quad (2)$$

$$y_3 = ad^2 \mp bd + c.. \quad (3)$$

The area under the parabola is given as

$$\int_{-d}^{d} (ax^2 + bx + c)dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-d}^{d}$$

$$\left[\frac{ad^3}{3} + \frac{bd^2}{2} + cd\right] - \left[\frac{-ad^3}{3} + \frac{bd^2}{2} - cd\right]$$

$$\Rightarrow n\ 2ad^3 + 2cd \cdots \quad (4)$$

From (1), (2) and (3); multiply equation multiply equation 2 by 4 we have

$$4y_2 = 4c \cdots \quad (5)$$

Add equations 1,2 and 5, we obtain $y_1 + y_3 + 4y_2 = 2ad^2 + 6c \cdots \quad (6)$

From (4) $\frac{2ad^2}{3} + 2cd \quad = \frac{d}{3}(2ad^2 + 6cd)$

Substituting this in equation 6.

Area $= \frac{d}{3}(y_1 + y_3 + 4y_2)$

If we take seven ordinates the expression for the area becomes

$$\frac{1}{3}(y_1 + 4y_2 + y_3) + \frac{1}{3}d(y_3 + 4y_4 + y_5) + \frac{1}{3}d(y_5 + 4y_6 + y_7)$$

$$= \frac{1}{3} d[y_1 + y_7 + 2(y_3 + y_5) + 4(y_2 + y_4 + y_6)]$$

Example 7:

Use Simpson's rule to find an approximation for the area under the curve

$$y = \frac{1}{x} blw \;\; x = 1 \text{ and } x = 2$$

Using 5 ordinates

x	1	1.25	1.5	1.75	2
y	1	0.80005	0.6667	0.5714	0.5

$$d = \frac{2 - 1}{4} = 0.25$$

$$\text{The area } d = \frac{d}{3}[y_1 + y_5 + 4(y_2 + y_4) + 2y_3]$$

$$= \frac{0.25}{3}[1 + 0.5 + 4(1.3714) + 2(0.667)]$$

$$\underline{\underline{= 0.6933}}$$

Example 8:

Find the x −coordinate of the centroid of the area bounded by $y = \sqrt{\sin x,} {}^{\pi}/_{2}$ and the $x - axis$ using Simpson's rule with five ordinate.

Solution

The centroid x −coordinate of the controid $\bar{x}$ is founded by

$$\bar{x} \int_0^{\pi/2} ydx = \int_0^{\pi/2} xydx$$

We first find $\int_0^{\pi/2} ydx = \int_0^{\pi/2} \sqrt{\sin x}\, dx$

area= $\frac{d}{3}[y_1 + y_5 + 2y_3 + 4(y_2 + y_4)]$

x	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$
$\sqrt{\sin x}$	0	0.686	0.8409	0.9611	1
$\sqrt{\sin x}$	0	0.2406	0.6604	1.1523	1.5708

$$d = \frac{\frac{\pi}{2} - 0}{4} = 0.392699$$

$$d/3 = 0.1309$$

$$Area = 0.1309[0 + 1 + 2(0.8409) + 4(1.5797)]$$

$$= 1.1781785$$

Now to determine

$$\int_0^{\frac{\pi}{2}} xydx = \int_0^{\frac{\pi}{2}} x\sqrt{\sin x}\, dx$$

$$Area' = \frac{d}{3}[y_1 + y_5 + 2y_3 + 4(y_2 + y_4)]$$

$$= \frac{0.1309}{3}[0 + 1.5708 + 2(0.6604) + 4(1.1323 + 0.2430)]$$

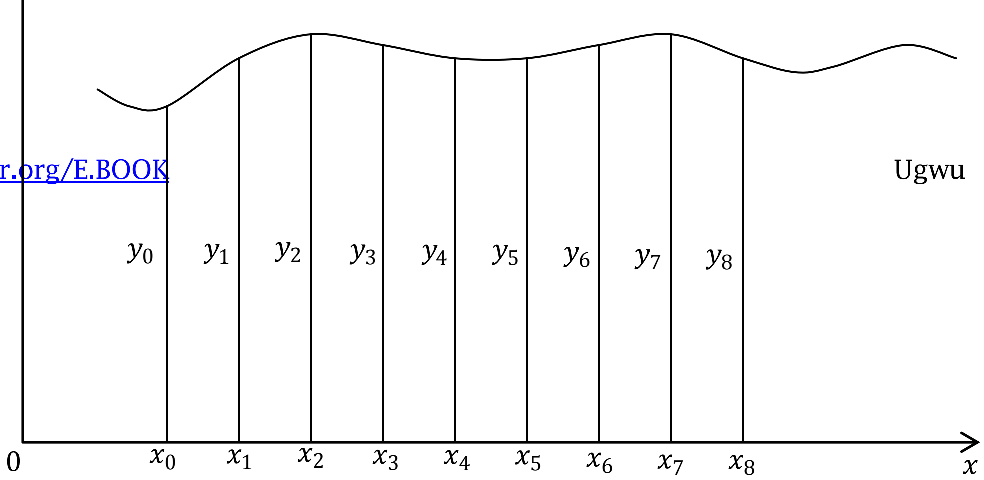
$$Area' = 1.0986175$$

$$\bar{x}Area = Area'$$

$$\therefore \bar{x} = \frac{1.0986175}{1.1781785} = 0.93$$

8.2.2 Trapezoidal Rule

An estimate of the area under a curve could be obtained by drawing in ordinates joining the tops of adjacent ones and calculating the areas of the trapeziums formed. For instance using seven ordinates estimate the area of a curve given as $y = f(x)$



$$The\ area = \frac{1}{2}d(y_1 + 2y_2 + y_3 + \frac{d}{2}(y_3 + 2y_4 + 2y_5 + 2y_6 + y_7$$

$$= \frac{1}{2}d[y_1 + y_7 + 2(y_2 + y_3 + y_4 + y_5 + y_6]$$

Use Trapezoidal rule to estimate the area under the curve $y = \frac{1}{x}$

From $x = 1\ to\ 2$ taking 6 ordinates.

Solution

$$d = (2 - \frac{1}{5} = 0.2$$

$$\int_1^2 ydx = \frac{0.2}{2}[y_1 + y_6 + 2(y_2 + y_3 + y_4 + y_5)]$$

x	1.0	1.2	1.4	1.6	1.8	2.0
y	1.0	0.8333	0.7143	0.6250	0.5556	0.5

$$Area = \int_1^2 \frac{1}{x}dx = \frac{0.2}{2}[1.0 + 0.5 + 2(0.8333 + 0.7143 + 0.6250 + 0.5556)]$$

$$= 0.69576$$

8.2.3 Numerical Harmonic Analysis

It is very important to observe that trapezoidal rule can be applied to estimate Fourier coefficient by considering one cycle of a periodic function of period in which we divide the trapezoidal sketch into n equal width strips, the width of each strip is $\frac{2\pi}{n}$.

For instance

$$\int_0^{2\pi} f(x)dx \cong d[y_0 + y_1 + y_2 + \cdots + y_{n-1}]$$

The values of $y_0, y_1, y_2, \ldots$ are often available as given table of values at regular intervals. If the function values are not given at regular intervals, the graph will be simply drawn of y against x and read off from the plotting.

Example

Evaluate $\int_0^{2\pi} f(x)dx$from the following set of function values

x^0	0	30	60	90	120	150	180	210	240	270	300	330	360
$f(x)$	1.4	1.6	2.0	2.1	1.9	1.1	0.4	0.4	0.7	0.6	0.5	1.0	1.4

The width of strip $d = \frac{\pi}{6}$

Substituting in

$$\int_0^{2\pi} f(x)dx \cong d[y_0 + y_1 + y_2 + y_3 + y_4 + \cdots + y_{n-1}]$$

$$= \frac{\pi}{6}\sum_{r=0}^{n-1} y_r = \frac{\pi}{6}[13.7] = 7.173$$

The accuracy of the result here depends on the number of strips taken.

If we consider Fourier series as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

Where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = 2\, mean\ value\ of\ f(x) over\ a\ period$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = 2\ \times mean\ value\ of\ f(x)\cos nx\ over\ a\ period$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = 2\ \times mean\ value\ of\ f(x)\sin nx\ over\ a\ period$$

Now what it means that if $f(x)$ is given as a set of evenly spaced function value, then one can evaluate each of these integrates by multiplying the mean value of $f(x)$, $f(x)\cos nx$, $f(x)\sin nx$ by 2 for successive values of n, over a complete cycle.

Now, we divide one complete cycle of the function into twelve equal width strips i.e. $\frac{2\pi}{12} = \frac{\pi}{6} = 30^0$ and tabulate the ordinates (function values) $y_0, y_1, y_2, y_3, y_4, y_5, \dots, y_{n-1}, \dots, y_n$.

The final boundary ordinate y_n is omitted since this is regarded as the ordinate of the next cycle.

Example

x^0	0	30	60	90	120	150	180	210	240	270	300	330	360
y_r	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_9	y_{11}	y_{12}
	1.4	1.6	2.0	2.1	1.9	1.1	0.4	0.4	0.7	0.6	0.5	1.0	1.4

$$a_0 = 2 \times \frac{Sum\ of\ the\ twelve\ ordinate}{12}$$

$$a_0 = \frac{1}{6}[y_0 + y_1 + y_2 + \dots + y_{11}]$$

$$a_0 = 2.283$$

To find a_1, we compute $\cos x$ and multiply by $f(x)$ i.e. we find $f(x)\cos x$.

x^0	0	30	60	90	120	150	180	210	240	270	300	330	360
$\cos x$	1.0	0.866	0.5	0	-0.5	-0.866	-1.0	-0.866	0.5	0	0.5	0.866	-
$f(x)\cos$	1.4	1.386	1.0	0	-0.95	-0.953	-0.4	-0.346	-0.35	0	0.2	0.866	-

$f(x)\cos$	1.4	0.8	-1.0	-2.1	-0.95	0.55	0.4	0.2	-0.35	-0.6	-0.25	0.5	-

Then

$$a_2 = 2 \times \frac{Mean\ value\ of\ f(x)\cos x}{12} = \frac{1}{6}(1.903) = 0.3172$$

To find a_2;

$$a_2 = 2 \times Mean\ value\ of\ f(x)\cos x\ \ over\ a\ period$$

$$= 2\frac{\sum f(x)\cos 2x}{12} = \frac{1}{6}(-1.40) = 0$$

$$\underline{\underline{= 0.2333}}$$

The reader can try this on his/her own $b_1 = 2x$ mean value of $f(x)\sin x$ over a period $-2\frac{\sum f(x)\sin x}{12}$

Questions

Using 5 ordinates find the value of $\int_0^1 \frac{dx}{1+x^2}$

(1). Using (a) Trapezoidal rule (b) Simpson's rule compare the two results (a) and (b)

(2). Evaluate $\int_0^1 e^{x^2}dx$ By Simpsons rule

Taking 10 intervals (11 ordinate)

(3). Use Simpson's rule with four strips to estimate $\int_0^{-2} \frac{1}{1+x^3}dx$ upto 3 sfg.

(4). A periodic function $y = f(x)$, of period 2π, is defined between $0 = x360^0$ by the following table of values.

x	0	30	60	90	120	150	180	210	240	300	330	360
y	3.0	4.0	4.6	4.8	3.6	2.8	2.2	1.1	0.6	1.6	2.0	–

Determine the Fourier series up to and including the 3rd harmonic and find the percentage of 3rd Harmonic:

Hint percentage of n^{th} harmonic is $\frac{A_n\ harmonic}{A_1 harmonic}$

$$A_1 = (a_1^2 + b_1^2)^{1/2}$$

$$\text{while} A_n = (a_1^2 + b_1^2)^{1/2}$$

8.2.4 Macluarin and Taylor

Theorem: These theorem expand $f(x+h)$ in terms of $f(x)$, powers of h and successive derivative

$$f(x+h) = f(x) + hf^I(x) + \frac{h^2}{2^1}f^{II}(x) + \frac{h^3}{3^1}f^{III}(x) + \frac{h^4}{4^1}f^{IV}(x) + \cdots$$

$$\cdots + \frac{h^n}{n!}f^n(x)$$

$f'(x)$ Denotes the first derivative while $f^n(x)$ denotes the n^{th} derivative etc.

If however, we let $h = x$ and set $x = 0$, we obtain Macluain's series.

That is $f(x) = f(0) + \frac{x}{1!}f^I(0) + \frac{x^2}{2!}f^{II} + \frac{x^3}{3!}f^{III}(0) + \frac{x}{4!}f^{IV}(0)$

$$+ \cdots + \frac{x^n}{n!}f^n(0).$$

Example 9

Evaluate $\int_0^1 x^{1/2} \cos x\, dx$

Solution

To solve the easily, we apply Macluarin's series expansion.

$$\times 100\% \qquad f(x) = \cos x, \quad f(0) = 1$$

$$f(x) = \cos x, \quad f(0) = 1$$

$$\therefore \cos x = \ f(0) + x\, f_{(0)}^I + \frac{x^2}{2!}f_{(0)}^{II} + \frac{x^3}{3!}f_{(0)}^{III} + \frac{x^4}{4!}f^{IV} + \frac{x^5}{5!}f_{(0)}^V + \frac{x^6}{6!}f_{(0)}^{VI}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}\cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{272} + \frac{x^8}{40320}$$

$$\int_0^1 x^{1/2}\left[| - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{272} + \frac{x^8}{40320}\cdots\right]dx$$

$$\left[\frac{2}{3}x^{3/2} - \frac{x^{9/2}}{7} + \frac{x^{11/2}}{152} - \frac{x^{15/2}}{272}\right]_0^1 = \frac{2}{43} - \frac{1}{7} + \frac{1}{132} - \frac{1}{2040}$$

$$\underline{\underline{\simeq 0.5309}}$$

TrialQuestion

If it is known that the Macluarin's expansion of

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots\cdots$$

And that $\sqrt{1 + sin\, x} = sin\frac{1}{2}x + cos\frac{1}{2}x$, show that Macluarin's expansion of $[1 + sin\, x]^{1/2}$ is

$$1 + x - \frac{x^2}{4.2!} - \frac{x^3}{4.3!} + \frac{x^4}{8.4!} - \frac{x^5}{16.5!} +$$

Example 10

If $f(x) = e^x\, sin\, x$; show that

$f^n_{(x)} = 2^{\frac{1}{2}n} e^x\, sin\left(x + \frac{\pi}{4}\right)$and use this with Macluarin's theorem to find an expansion for $f(x)$ in ascending power of x as far as the x^6 term

$$f'(x) = e^x\, sin\, x$$

$$f(x) = e^x\, sin\, x + e^x\, \sigma s\, x$$

Let $1 = a\cos\varepsilon$ and $1 = a\, sin\, \varepsilon$, then $\frac{a\sin\varepsilon}{a\cos\varepsilon} = 1$

$$\tan\varepsilon = 1; \quad \varepsilon = tan^{-1}1 = \frac{\pi}{4}.$$

$$a^2cos^2\varepsilon + a^2sin^2\varepsilon = 1^2$$

$$a^2(cos^2\varepsilon + a^2sin^2\varepsilon) = 2$$

$$a = \sqrt{2}$$

$$\therefore f^{'(x)} = a\, e^x \sin x \cos\varepsilon + ae^x \sin(x + \varepsilon)$$

$$f^I(x) = \sqrt{2}e^x \sin\left(x + \frac{\pi}{4}\right)$$

Similarly $f^n(x) = \sqrt{2}^n e^x\, sin\left(x + \frac{\pi}{4}\right)$

$$f^{II}(x) = \sqrt{2}^2 e^x\, sin\left(x + \frac{2\pi}{2}\right)$$

$$f^{III}(x) = \left(\sqrt{2}\right)^3 e^x \, sin\left(x + \frac{3\pi}{4}\right)$$

$$f^{IV}(x) = \left(\sqrt{2}\right)^4 e^x \, sin\left(x + \frac{4\pi}{4}\right)$$

$$f^{V}(x) = \left(\sqrt{2}\right)^5 e^x \, sin\left(x + \frac{5\pi}{4}\right)$$

$$f^{VI}(x) = \left(\sqrt{2}\right)^5 e^x \, sin\left(x + \frac{6\pi}{4}\right)$$

$$f(0) = 0$$

$$f^{I}(0) = \sqrt{2}.\frac{1}{\sqrt{2}} = 1$$

$$f^{II}(0) = 2$$

$$f^{III}(0) = \sqrt[2]{2}.\frac{1}{\sqrt{2}} = -2$$

$$f^{IV}(0) = 0$$

$$f^{V}(0) = \sqrt[4]{2}.\frac{1}{\sqrt{\sqrt{2}}} = -4$$

$$f^{VI}(0) = 8x(-1) = -8$$

$$\therefore f(x) = 0 + x^2 + \frac{x^3}{3} - \frac{5}{30} - \frac{x^6}{90}$$

Example 11

If $y = e^{4x} \cos 3x$, prove that

$y' = 5e^{4x} \cos(3x{+}\propto)$,where$tan \propto = {}^{2}/_{4}$, use Macluarin's theorem to find the expansion of y in ascending power's of x as far as term in x^3.

Solution

$$y = e^{4x} \cos 3x \cdots \qquad (1)$$

$$y' = 4e^{4x} \cos 3x - 3e^{4x} \sin 3x \cdots$$

Now let $4 = a \cos \propto$ and $3 = a \sin \propto$

$$\tan \propto = {}^{3}/_{4}$$

$$a = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

Substituting in equation 2, we have

$$y' = a\,e^{4x}\cos 3x \cos \propto - a3e^{4x}\, sin\, 3x\, sin \propto$$

$$= 3e^{4x}(\cos 3x \times \propto) = 5e^{4x}\cos(3x + 37^0)$$

Where $a = 5$; $\propto = 37^0$

$$y(0) = 1$$

$$y'(0) = 5 \times 0.79968$$

$$y''^{(0)} = -4.18$$

$$\therefore f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f''(0)$$

$$= 1 + 4x + \frac{7x^2}{2} - \frac{44x^3}{6}$$

8.2.5 Euler's Method

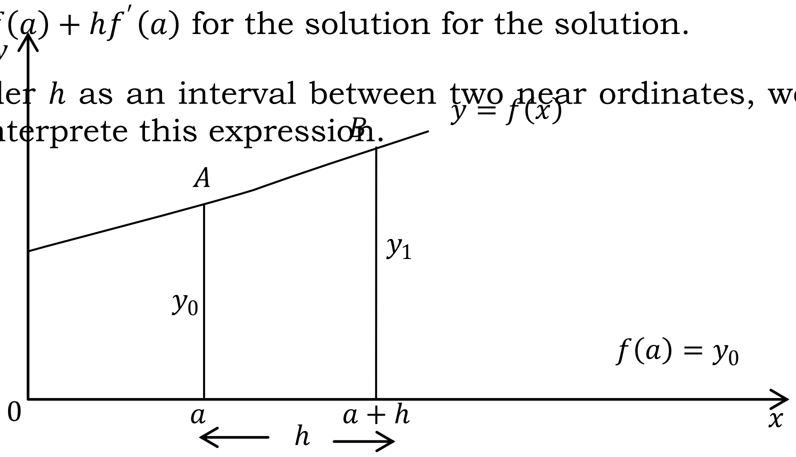
This method is one of the simplest numerical methods for solving first –order differential equations. It involves truncation of Taylor's series after the second term.

That is for Taylor's series of the form;

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f''^{(a)} + \cdots$$

We use $f(a + h) \simeq f(a) + hf'(a)$ for the solution for the solution.

Now we can consider h as an interval between two near ordinates, we can use graph to interprete this expression.



We can rewrite the above approximated expression as $y_1 = y_0 + h(y')0$ such that if now know y_0, h and $(y')0$, we can compute y_1, an approximate value for the function value at B.

Generally, one can write

$$y_{r+1} = y_r + hf(x_r, y_r).$$

$$(y = y - 2x)$$

Example 12

Given that $f(x,y) = y - 2x$ when $y(0) = 3$ where $x_0 = 0,\ x_0 = 0.2, x_2 = 0.4, x_3 = 0.6$

$$y_{r+1} = y_r + 0.2\ (y_r - 2x_r)$$

$$\text{for } r = 0$$

$$y_1 = y_0 + 0.2\ (y_0 - 2x_0)$$

$$= 3 + 0.2(3 - 0) = 3.6$$

$$\text{for } r = 1$$

$$y_2 = y_1 + 0.2\ (y_1 - 2x_1) x_1 = 0.2$$

$$= 3.6 + 0.2(3.6 - 2 \times 0.2)$$

$$y_2 = 4.24$$

$$\text{for } r = 2$$

$$y_3 \neq y_2 + 0.2\ (y_2 - 2x_2)$$

$$= 4.24 + 0.2(4.24 - 2 \times 0.4) = 4.928$$

Example 13

Find $y(2.5)$ for $IVPy = 3y$ using Euler method with $h = 0.1$

Solution

$$x_0 = yx_1 = 2.1, x_2 = 2.2, x_3 = 2.3, x_4 = 2.4, x_5 = 2.5$$

$$y_{(0)} = y(x_1), y_{(2)} = 10$$

$$f(k, y) = 3y$$

$$y_{r+1} = y_r + hf\ (x_r, y_r)$$

$$y_{r+1} = y_r + 0.1\ (3y_r$$

$$\text{for } r = 0$$

$$y_1 = 1.3\ y_0 = 1.3\ (10) = 13$$

$$\text{for } r = 1$$

$$y_2 = 1.3\, y_1 = 1.3\ (1.3) = 16.9$$

for $r = 2$

$$y_3 = 1.3\, y_2 = 1.3\ (16.9) = 21.97$$

$$y_4 = 1.3\, y_3 = 1.3\ (21.97) = 28.561$$

$$y_5 = 1.3\, y_4 = 1.3\ (28.97) = 37.1293$$

$$\therefore y(1.5) = y(x_5) = 37.1293.$$

Example 14

Obtain a numerical solution of the equation

$y^{'} = 1 + x - y$ with the initial condition from $x = 1$ to 2 at constant intervals of $x = 0.2$

Solution.

We first write

$$yx_1 =, y_r + 0.2(r + x_r - y_r)$$

for $r = 0$,we have

$y_1 = \ y_0 + 0.2\ (1 + x_0 - y_0)$

Now as given $x_0 = 1$ at $y_0 = 2$

$\therefore y_1 = 2 + 0.2[(1 + 1)^{-2}]$

$y_1 = 2$

for $r = 1$

$y_2 = y_1 + 0.2\ [(1 + x_1) - y_1]$

$y_2 = 2 + 0.2\ [1 + 1.2 - 2] = 2 + 0.04$

$y_2 = 2.04.$

for $r = 2$

$y_3 = y_2 + 0.2[1 + x_2 - y_2]$

$y_3 = 2.04 + 0.2(1 + 1.4 - 2.04)$

$y_3 = 2.04 + 0.2(2.36) = 2.112$

When $r = 3$

$y_4 = y_3 + 0.2[1 + x_3 - y_3] y_4 = 2.112 + 0.2[1 + 1.6 - 2.112]$

$y_4 = 2.2096$

for $r = 4$

$y_5 = y_4 + 0.2[1 + x_4 - y_4]$

$$2.2096 + 0.2[1 + 1.8 - 2.2096]$$

$$= 2.32768$$

when $r = 5$

$y_6 = y_5 + 0.2\ [(1 + x_5) - y_5]$

$$= 2.32768 + 0.2[1 + 2.2 - .32768]$$

$$= 2.3411264$$

These values can be plotted.

8.2.6 Runge Kutta Method

This method for solving differential equations especially first order differential equation is widely used with a reasonable high degree of accuracy. It involves a step by-step process where a table of function values for a range of values of x is accumulated. This requires several intermediate calculations in general, if we wish to solve.

$y^{'} = f(x, y)$with initial condition $y = y_0$ at $x = x_0$ for a range of values of $x = x_0(h)x_n$

Starting as usual with $x = x_0; y = y_0; y^{'} = (y^{'})_0$ and h, we have

$$x_1 = x_0 + h$$

To calculate $y_1$requires 4 intermediate stages

$$k_1 = hf(x_0, y_0) = h(y')_0$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$k_4 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_3\right)$$

The increase in $y - values\ from\ x = to\ x = x_1\ is$

$$\Delta y_0 = \frac{1}{6}[k_1 = 2k_2 + 2k_3 + k_4].$$

$$y_1 = y_0 + \Delta y_0$$

Example 15

$$y' = x = y$$

Using $y = 1$ when $x = 0$ i.e $y(0) = 1$.

Determine the function values of y for $x = 0(0.1)0.5$

Solution.

$$x_0 = 0, y_0 = 1, (y')0 = 1, h = 0.1$$

$$(y')0 = (1 + 0)$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$k_1 = h(y')0 = k_1 = 0.1 \times 1 = 0.1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h\right), y_0 + \frac{1}{2}k_1\right)$$

Now $x_0 + \frac{1}{2}h = 0 \times 0\, 1/2\, (0.1) = 0.05$

$$y_0 + \frac{1}{2}k_1 = 1 + \frac{1}{2}(0.1) = 0.05$$

$$k_2 = 0.1[0.05 + 1.05] = 0.11$$

$$k_3 = hf\left[x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right]$$

Here $x_0 + \frac{1}{2}h$ and y_0 remain the same

$$x_0 + \frac{1}{2}h = 0.05, \qquad y_0 + \frac{1}{2}k_2 = 1 + \frac{1}{2}(0.11)$$

$$= 0.055 + 1 = 1.055$$

$$k_3 = 0.1[0.05 + 1.055] = 0.1105$$

For k_4

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$y_0 + \frac{1}{2}k_3 = 1 + 1/2 + (0.1105)$$

$$= 1.1105$$

$$\therefore k_4 0.1[0.1 + 1.1105] = 0.12105$$

Since k_1, k_2, k_3, k_4 have been obtained, Δy_0 can also obtained thus, $\Delta y_0 = \frac{1}{6}[k_1 + k_2 + k_3 + k_4]$

$$= \frac{1}{6}[0.1 + 0.11 + 0.1105 + 0.12105]$$

$$\Delta y_0 = 0.110342$$

Now $y_1 = y_0 + \Delta y_0 = 1 + 0.110342 = 1.110342$

Finally, from the equation,

$$(y')_1 = x_1 + y_1$$

Where $x_1 = 0.1, y = 1.110342$

$$(y')_1 = x_1 + y_0 = 0.1 + 1.110342 = 1.210342$$

Now we repeat the same process this second sage considering as our $x_0 = 0.1$, and y_1 as y_0

$$\therefore (y')_1 = x_1 + y_1$$

$$(y')0 = 0.1 + 1.110342 = 1.210342, h = 0.1$$

$$k_1 = h(y')0 = 0.1 + 1.20342 = 0.121034$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$x_0 = 0.1, h = 0.1\text{:}\ \ x_0 + \frac{1}{2}h = 0.1 + 0.05 = 0.15$$

$$y_0 = 1.110342, \qquad y_0 + \frac{1}{2}k_1 = 1.110342 + \frac{1}{2}0.121034$$

$$k_2 = 0.1[0.15 + 1.170859] = 0.132 = 0.12986$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$y_0 + \frac{1}{2}k_2 = 1.11.342 + \frac{1}{2}(0.132086) = y_0 + \frac{1}{2}k_2 = 1.176385$$

$$k_3 = 0.1[0.15 + 1.176385] = 0.132639$$

$$k_4 = hf\ (x_0 + h, \qquad y_0 +\ k_4) = 0.1(0.2 + 1.242981)$$

$$k_4 = 0.144298$$

Again we can obtain $Ay_0 = \frac{1}{6}(k_1 + k_2 + 2k_3 + 2k_4)$

$$= \frac{1}{6}(0.121034 + 2(0.132986) + 2(0.32639) + 0.144298$$

$$Ay_0 = 0.132464.$$

$$y_1 = y_0 + Ay_0 = 1.110342 + 0.132464$$

$$= 1.242806$$

$$(y')_1 = x_1 + y_1 = 0.2 + 1.242806$$

$$= 1.442806.$$

Third stage can be considered and go on. That is the iterative process can go on as many as wanted.

Table of results can be formed accordingly.

x	y'	y
0	1.0	1.0
0.1	1.110342	1.210342
0.2	1.24208	1.442806
.	.	.
.	.	.
.	.	.

Second order differential equations second order differential equations can be solved numerically by using Euler's method and Rung-Kuta method.

Euler's method involved direct application of the truncated form of Taylor's series which is easy but not accurate to a reasonable degree.

In this case,

$$y_1 = y_0 + \frac{1}{2} + h(y')_0 + \frac{h^2}{2!}(y'')_0$$

$$(y')_1 = (y')_0 + h(y'')_0.$$

If $x_0, = y'_0$ and $h(y'')_0$ are computed, the approximate value of y_1 at $x_1 = x_0 + h$ can be obtained.

Example 16

Solve$y'' - xy' + y$ for $x = 0(0.2)1.0$ given that $x = 0, y = 1$ and $y' = 0$.

Solution

$$x_0 = 0, y_0 = 1, (y')_0, h = 0.2$$

From the differential equation;

$$(y'')_0 = 0 + (y')_0 + 1 = 1$$

Using

$$y_1 \neq y_0 + h(y')_0 + \frac{h^2}{2!}(y'')_0$$

$$y_1 = 1 + (0.2)(y')_0 + \frac{(0.2)^2}{2!}(1)$$

$$= 1 + 0.2 \times 0 + 0.02 = 1.02.$$

$$(y')_1 = (y')_0 + h(y'')_0 = 0 + 0.2(1) = 0.2.$$

Now $y'' = xy' + y \equiv$ and $x_1 = 0.2$

$$(y'')_1 = x_1(y')_1 + y_1 = 0.2(0.2) + 1.02.$$

$$(y'')_1 = 0.0441.02.$$

Now we have $x_1 = 0.2, +y_1 = 0.2y' = 1.02, (y')_1 = 02.$

$$(y'')_1 = 1.06.$$

In this method, for us to proceed to second stage, one use these result as a starter.

$$x_0 = 0.2, \;\; y_0 = 10.2, y'_0 = 02, \;\; (y'')_0 = 1.06$$

$$x_1 = 0.4$$

Thus,

$$y_1 = y_0 + h(y')_0 + \frac{h^2}{2!}(y'')_0$$

$$y_1 = 1.02 + 0.2(0.2) + \frac{(0.2)^2}{2!}(1.06) = 1.0812$$

$$y_1 = 1.0812$$

Again, $(y')_1 = (y')_0 + h(y'')_0 = 0.2 + 0.2 \times 1.06 = 0.412.$

$$\therefore \quad \text{for } (y'')_1 = x_1(y')_1 + y_1$$

$$= 0.4(0.412) + 1.0812$$

$$(y'')_1 = 1.2460$$

3rd stage: In this stage, we take.

$$x_0 = 0.4,\ \ y_0 = 1.0812,\ \ (y'_0) = 0.412$$

$$(y'')_0 = 1.2460$$

And repeat the same process again

$$y_1 = y_0 + h(y')_0 + \frac{h^2}{2!}(y'')_0$$

$$= 1.0812 + 0.2 \times 0.412 + \frac{(0.2)^2}{2!}1.2460$$

$$= 1.18852$$

$$(y')_1 = (y')_0 + h(y'')_0 = 0.412 + 0.2(1.246).$$

$$(y')_1 = \quad 0.412 + 0.024920$$

$$= 0.66120$$

Using $(y'')_1 = x_1(y')_1 + y_1$

$$(y'')_1 = 0.6(0.66120) + 1.18852$$

$$= 0.396720 + 1.18852$$

$$= 1.58524$$

For 4th stage,

$$y_1 = y_0 + h(y')_0 + \frac{h^2}{2}(y'')_0$$

First we obtain $(y')_1 = (y')_0 + h(y'')_0$

$$(y')_0 = 0.661200$$

$$(y')_0 = 0.66120 + 0.2(1.58524)$$

$$= 0.66120 + 0.317048$$

$$= 0.978248$$

$$y_1 = y_0 + h(y')_0 + \frac{h^2}{2!}(y'')_0$$

$$y_1 = 1.18852 + 0.2(0.978248) + 0.02(1.58524)$$

$$= 1.4158744$$

$$(y'')_1 = x_1(y')_1 + y_1$$

$$= 0.8(0.978248) + (1.4158744)$$

$$= 2.21413424$$

The reader can go ahead for the 5th stage of the computation. In this method, the error increases rapidly as the computation progresses from the initial values. The reason for this is as a result of the truncation of the Taylor's series on which the method is based on.

CHAPTER 9

SPECIAL AND DIRAC DELTA FUNCTIONS

9.1.0 Integral Functions

One of the preliminary mathematical courses in sciences and engineering that is necessary as a prerequisite for further mathematical courses in sciences and engineering is integration. Various techniques for the analytic evaluation of definite integrals are presented here. Here in this text, we did not discuss integration as a matter of fact.

This topic is of importance problems in science and engineering cannot be found in terms of elementary functions such as cosine, sine etc. But rather may be found in terms of an infinite convergent series terms which may again constitute a new function often called Tabulated function. Thus in this chapter, we discuss some of these tabulated function, beta and elliptic function. Interesting expect of this concept is that it involves fundamental recurrence relation which has be earlier presented to us in series solution of differential equation

9.1.1 Gamma function

Gamma function is defined for Re N>o by the integral

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx \qquad 9.1$$

Where x is a parameter and N is any number

From equation 1, we write

$$\Gamma(n+1) \int_0^\infty e^{-x}\, x^{n} dx \qquad 9.2$$

Here we apply integration by part to obtain

$$\Gamma(n+1) = [-x^n e^{-x}]_0^\infty + n\int_0^\infty e^{-x}\, x^{n-1} dx$$

$$= 0 + n\int_0^\infty e^{-x}\, x^{n-1} dx$$

$$\Gamma(n+1) = n\int_0^\infty e^{-x}\, x^{n-1} dx$$

From our definition as in equation 1,

$$\Gamma(n+1) = \int_0^\infty e^{-x}\, x^{n-1} dx$$

Thus

$$\Gamma(n+1)\Gamma(n) \qquad 9.3$$

Equation (3) forms the fundamental basis of the recurrence relation for gamma function. With this, we can generate other form of results such as

$$\Gamma(n+1) = n(n-1)(n-2)(n-3\cdots\ etc \qquad 9.4$$

Now for $n = 1$

$$\Gamma(1) = \int_0^\infty e^{-x}\, x^{1-1} dx$$

$$= \int_0^\infty e^{-x}\, x^0 dx = [-e^{-x}]_0^\infty = 1$$

Therefore we have that $\Gamma(1) = 1\ and\ \Gamma(n+1) = n!$

Provided that n is positive integer.

We can simply use the recursion relation such as;

$$\Gamma(n+1) = n\Gamma\text{n} \text{ to solve for}$$

$$\Gamma(3), \qquad \Gamma(5), \qquad \Gamma(7)$$

For $\Gamma(3) = \Gamma(2+1) = 2!\,\Gamma(1) = 2! = 2$

For $\Gamma(5) = \Gamma(4+1) = 4!\,\Gamma(1) = 4! = 24$

$$\Gamma(7) = \Gamma(6+1) = 6!\,\Gamma(1)\ 6! = 720$$

The recurrence relation can also be written in reverse form

$$\Gamma(n+1) = n\Gamma(\text{n})$$

$$\therefore \Gamma(\text{n}) = \frac{\Gamma(n+1)}{\text{n}}$$

For instance

$$\Gamma(6) = \frac{\Gamma 6+1}{6} = \frac{\Gamma 7}{6} = \frac{720}{6} = 120$$

However, we note that n is not defined for any negative integer

Therefore when $n = -\frac{1}{2,}$ we use the reversed recurrence relation to solve the problem.

Example

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma 1 - 1}{-\frac{1}{2}} = -2\,\Gamma\frac{1}{2} \Rightarrow -2\frac{\Gamma\left(\frac{1}{2}+1\right)}{\frac{1}{2}}$$

$$= -2 \times 2\,\Gamma\frac{3}{2}$$

The expression $\Gamma\frac{3}{2}$ is provided in a table and should be obtained from there. $\Gamma\frac{3}{2} = 0.8862$as in the table

Thus

$$\Gamma\left(-\frac{1}{2}\right) = -4 \times 0.8862$$

Example 2

Evaluate $-1.3!$or $\Gamma(-1.3)$

$$\Gamma(-1.3) = \Gamma - 0.3$$

Now use the reversed recurrence relation

$$\Gamma(n) = \Gamma\frac{n+1}{n} = \Gamma\frac{1-0.3}{-0.3} = \Gamma\frac{0.7}{-0.3}$$

We repeat again $\Gamma 0.7$ the same process

$$\Gamma 0.7 = \Gamma\frac{(1+0.7)}{0.7} = \Gamma\frac{1.7}{0.7}$$

$$\Gamma - 1.3 = \frac{\Gamma 1.7}{-0.3 \times 0.7} = \frac{\Gamma 1.7}{0.7}$$

$\Gamma 1.7$is to be obtained from a table and it is 0.9086

$\Gamma(-1.3) = \frac{0.9086}{-021} = -4.326$

Evaluation of integrals related to $\Gamma(n)$.

By definition

$$\Gamma(n) = \int_0^a e^{-x}\, x^{n-1} dx \ldots\ldots\ldots\ldots (5)$$

$$n > 0$$

We not that

$$\int_0^\infty e^{-x} x^2 \; dx = \Gamma 3 = 2! = 2$$

To solve for

$$\int_0^a e^{-x/2}\, x^{1.4}$$

We first of all use a simple substitution to make it relate to substitution to make it relate to equation (4)

Here let $x/_2 = u$

$$x = 2u$$

$$\frac{dx}{du} = 2; \;\; dx = 2du$$

Now we substitute the back into the original equation to obtain

$$I = \int_0^\infty e^{-u}(2u)^{1.4}.2du$$

$$= \int_0^\infty 2^{2.4}\, U^{1.4}\, du$$

$$\Rightarrow I = 2^{2.4}\Gamma(1.4+1) = 2^{2.4}\Gamma(2.4).$$

Examples 3

Evaluate

$$I = \int_0^\infty x^{\frac{1}{2}}\, e^{-x^2} dx$$

Solution

Let $u = x^2; \;\; x = u^{\frac{1}{2}}$

$$du = 2x\, dx$$

To obtain the range, we find the limit

$$\lim_{x\to 0} u = 0$$

From the change of variable,

$$x \to \infty, \qquad u = \infty$$

$$\therefore \; x^{\frac{1}{2}} = \left(u^{\frac{1}{2}}\right)^{\frac{1}{2}} = u^{\frac{1}{4}}$$

$$\therefore I = \int_0^\infty u^{\frac{1}{4}}\, e^{-u}\, \frac{du}{2x} = \int_0^\infty \frac{u^{\frac{1}{4}} e^{-u}}{2u^{\frac{1}{2}}}\, du$$

$$= \frac{1}{2}\int_0^\infty u^{\frac{1}{4}}\, e^{-y}\, du$$

If let $n = \frac{3}{4}$, then

$$I = \frac{1}{2}\int_0^\infty u^{n-1}\, e^{-u}\, du;$$

$$= \frac{1}{2}\Gamma\frac{3}{4}$$

The value of $\Gamma\frac{3}{4}$ can be obtained from a table

We use the original definition of

$$\Gamma(n) = \int_0^0 x^{n-1} e^{-x}\, dx$$

To obtain

$$\Gamma = \int_0^\infty \mathrm{x}^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{x}} dx \qquad 9.6$$

Let $x = u^2;\;\; dx = 2u\, du$

Then

$$\Gamma\frac{1}{2} = 2\int_0^\infty e^{-u^2}\, du$$

Where

$$\int_0^\infty e^{u^2}\, du$$

Cannot be normally obtained

$$I^2 = 2\int_0^\infty e^{-u} d \cdot 2\int_0^\infty e^{-v}\, dv$$

$$= 4\int_0^\infty e^{-(u^2+v^2)}\, du\, dv$$

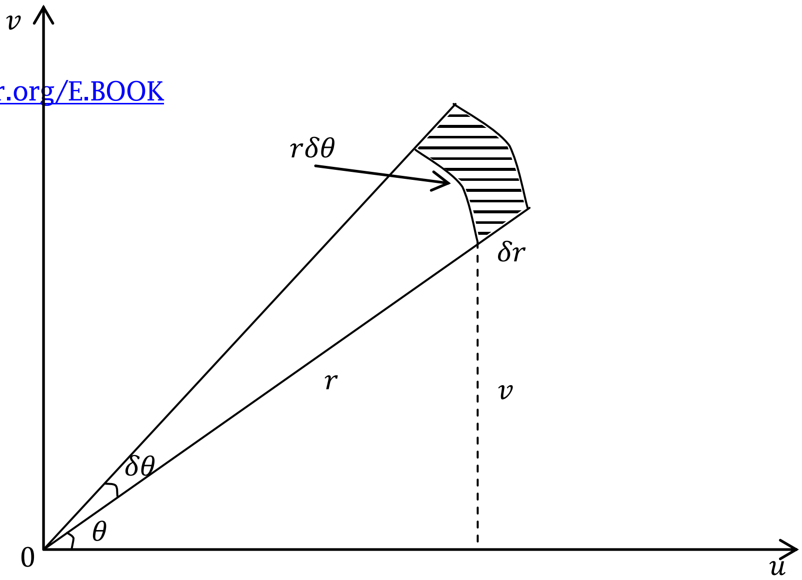
Now we change to polar coordinates

$$u = r\cos\theta\,, \qquad v = r\sin\theta$$

Where $u^2 + v^2 = r^2$

$$du\; r\delta\theta\delta r \qquad 9.7$$

For the integration to cover the same region, we consider 2 limits for $r; r = 0\;\; to\;\; r = \infty$ and limits for θ are $\theta = 0\;\; to\;\; \theta = \frac{\pi}{2}$



$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2}\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{e^{-r^2}}{2}\right]_0^{\infty} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\right) d\theta = \left[\frac{\theta}{2}\right]_0^{\pi} = \frac{\pi}{4}$$

$$\therefore I = \frac{\sqrt{\pi}}{2}$$

$$\therefore \int_0^{\infty} e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$$

Now having known already that

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-u}\, du$$

And

$$\int_0^{\infty} e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{1}{2}\right) = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

From the recurrence relation we can obtain $\Gamma\left(\frac{3}{2}\right), \Gamma\left(\frac{5}{2}\right)$ etc for instance in the case of $\Gamma\left(\frac{5}{2}\right)$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\,\Gamma\frac{1}{2} = \frac{1}{2}\ \Gamma\pi$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\ \Gamma\left(\frac{1}{2}\right) = \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} = \frac{3\sqrt{\pi}}{4}$$

9.1.3 Beta Function

Beta function denoted by $B(m,n)$ is defined as

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad 9.8$$

And converges for $m > 0$and $n > 0$. It can also be written as

$$B(n,m) = \int_0^1 x^{n-1}(1-x)^{n-1}dx \qquad 9.9$$

If we take $x = \sin^2\theta$, then $dx = 2sin\,\theta\cos\theta\,d\theta$ and when substituted into equation 9, it become

$$B(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta\, cos^{2n-1}\theta\, d\theta \qquad 9.10$$

The internal range emanates from considering $x = 0$ when $\theta = 0$ and $x = 1$ which make $\theta = \pi/2$ recalling our knowledge of reduction formulae

$$\int_0^{\pi/2} \sin^n\theta\; d\theta = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-1}\theta\, d\theta \qquad 9.11$$

$$\int_0^{\pi/2} \cos^n\theta d\theta = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2}\theta\, d\theta \qquad 9.12$$

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\, d\theta = \frac{m-1}{m+1}\int_0^{\pi/2} \sin^{m-2}\theta\cos^n\theta d\theta \qquad 9.13$$

If $I_{m,n}$is used to denote the integral on the left hand side, the our result becomes

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} \qquad 9.14$$

If we apply equation (14) to the integral, we have

$$\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

$$= \frac{(2m-1)-1}{(2m-1)+(2n-)}\int_0^{\pi/2} \sin^{2m-3}\theta\cos^{2n-1}\theta\, d\theta$$

$$= \frac{m-1}{m+n-1}\int_0^{\pi/2} \sin^{2m-3}\theta \cos^{2n-1}\theta \, d\theta \qquad 9.15$$

if we apply the same principle with the right hand integral we obtain

$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$ 9.16

$$= \frac{m-1}{m+n-1}\cdot\frac{(2n-1)-1}{2m-3)+(2n-1)}\cdot\int_0^{\pi/2} \sin^{2m-3}\theta \cos^{2n-3}\theta \, d\theta$$

$$\Rightarrow B(m,n) = \frac{(m-1)(n-1)}{(m+n-1)(m+n-2}\cdot B(m-1,n-1) \qquad 9.17$$

equation 17is a reduction formula for $B(m,n)$. The process can be repeated again and again as required

Example

Evaluate $B(5{,}4)$.

$$\frac{(4)(3)}{(8),(7)}B(4{,}3)$$

$$B(4{,}3) = \frac{(3)(2)}{(6)(5)}B(3{,}2)$$

$$B(3{,}2) = \frac{(2)(1)}{(4)(3)}\ \mathrm{B}(2{,}1)$$

15. $\Longrightarrow B(5{,}4) = \frac{(4)(3)(3)(2)(2)(1)}{(7)(7)(6)(5)(4)(3)}\ \mathrm{B}(2{,}1)$

$B(2{,}1)$can be evaluated using the integral

$$B(2{,}1) = 2\int_0^{\frac{\pi}{2}} sin^3\,\theta\; cos\theta\; d\theta = 2\left[\frac{sin^4\,\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{1}{2}$$

$$\therefore B(5{,}4) = \frac{(4)(3)(3)(2)(2)(1)}{(8)(7)(6)(5)(4)(3)}\frac{1}{2}$$

$$= \frac{4!\,6}{8!}$$

Example

Evaluate $B\left(\frac{1}{2},\frac{1}{2}\right)$

Solution:

Using

$$B(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\, cos^{2n-1}\theta\, d\theta$$

Since $m = \frac{1}{2}0$ and $n = \frac{1}{2}$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} sin^0\theta \cos\theta\, d\theta$$

$$\Rightarrow B\left(\frac{1}{2},\frac{1}{2}\right) = 2 = \int_0^{\frac{\pi}{2}} d\theta = 2\,[\theta]_0^{\frac{\pi}{2}}$$

$$= \pi$$

If m and n are positive integers

$$(Bm,n) = \frac{(m-n)!\,(n-1)!}{(\mathrm{m}+\mathrm{n}-1)!}$$

And already we have known that

$$n!\,\Gamma(\mathrm{n}+1) \text{ for } \mathrm{n} > o$$

Then

$$(\,m-1)! = \Gamma(\mathrm{m}) \text{ and } (\mathrm{n}-1)! = \Gamma(\mathrm{n})$$

And also

$$(m+n-1)! = \Gamma(\mathrm{m}+\mathrm{n})$$

$$\Rightarrow B(m,n) = \frac{(m-1)!\,(n-1)!}{(\mathrm{m}+\mathrm{n}-1)!} = \frac{\Gamma(m)\Gamma(\mathrm{n})}{\Gamma(\mathrm{m}+\mathrm{n})} \qquad 9.18$$

This holds even when m and n are not necessary integer. This is relation between the gamma and beat function.

9.1.3 Elliptic Functions

Elliptic function provides a means of evaluating range of definite integrals that can be converted by substitution into certain standard form. For instance

$$\int_0^1 \frac{\mathrm{dx}}{\sqrt{(1-2\mathrm{x}^2)(4-3\mathrm{x}^2)}} \qquad 9.19$$

Is an elliptic function

Elliptic function is of two kinds

$$F_1K(x)\int_0 \frac{d\theta}{\sqrt{1+K^2 sin^2\theta}} \qquad 9.20$$

Where $0 \leq \theta \leq \frac{\pi}{2}$ *and* $0 < K$

$$F_2K\left(\frac{\pi}{2}\right) = 9.21$$

Example

Evaluate

$$I = \int_0^{\pi/6} \frac{d\theta}{\sqrt{1-4\sin^2\theta}}$$

where$K^2 = 4,\ K = 2$

and this does not agree with the standard requirement that $0 < K < 1$ and the needed modification in order to meet up.

Let $4\,sin^2\theta = \sin 2\,\phi$

$\therefore 2\sin\theta = \sin\phi;$

$\sin\theta = \frac{1}{2}\sin\phi$

$\sin^2\theta = \frac{1}{4}\sin^2\phi$

we different both sides

$2\,cos\,\theta\,d\theta = \cos\phi\,d\phi$

$d = \frac{cos\theta d\phi}{2cos\theta}$

from$2sin\theta = sin\phi$;when $\theta = 0, \phi = 0$ and when $\theta = {}^{\pi}/_{6,}\ \phi = {}^{\pi}/_{2}$

substituting all the values into the original equation;

$$\int_0^{\pi/6} \frac{d\theta}{\sqrt{1-4sin^2\theta}} = \int_0^{\pi/2} \frac{1}{\sqrt{1-sin^2\phi.}} \frac{cos\phi d\phi}{2cos\theta}$$

since

$$sin^2\theta = \frac{1}{4}sin^2\,\phi, 1 - cos^2\theta = \frac{1}{4}sin^2\phi$$

$$\therefore cos\theta = \sqrt{1-\frac{1}{4}sin^2\phi}$$

$$\therefore \int_{\theta}^{\pi/2} {}^{1}/_{2} \frac{1}{cos\phi} \frac{cos\phi d\phi}{\sqrt{1-\frac{1}{4}sin^2\phi}} = \frac{1}{2}\int_{0}^{\pi} \frac{d\phi}{\sqrt{1-\frac{1}{4}sin^2\phi}}$$

which is now in a standard form

Thus

$$\mathrm{I} = \frac{1}{2}\mathrm{F}\left(\frac{1}{2,}{}^{\pi}/_{2}\right) = \frac{1}{2}\mathrm{K}\left(\frac{1}{2}\right)$$

From the standard elliptical function table, $\mathrm{K}\left(\frac{1}{2}\right) = 1.6858$

Example:

Evaluate:

$$I = \int_{0}^{\sqrt{3}/4} \sqrt{\frac{2-x^2}{1-4x^2}}dx$$

Solution

The first procedure to the solution is to convert the expression into the standardize form $\sqrt{\frac{1-K^2U^2}{1-U^2}}du$

Taking the denominator we put $4x^2 = U^2 \therefore 2x = u; 2dx = du = dx = \frac{du}{2}$

Limits: when $x = 0,\ u = 0$and when$x = \frac{\sqrt{3}}{4},\ u = \frac{\sqrt{3}}{2}$

From the numerator

$$2 - x^2 = 2 - \frac{u^2}{4}$$

Substituting back into the original integral at becomes

$$I = \int_{O}^{\frac{\sqrt{3}}{2}} \sqrt{\frac{2-\frac{u^2}{4}}{1-u^2}} \cdot \frac{du}{2} = \int_{O}^{\frac{\sqrt{3}}{2}} \sqrt{\frac{\frac{2}{4}-\frac{u^2}{16}}{1-u^2}}\, du$$

$$= \int_{0}^{\frac{\sqrt{3}}{4}} \frac{1}{\sqrt{2}} \sqrt{\frac{1-\frac{u^2}{8}}{1-u^2}}du$$

i.e. $k^2 = \frac{1}{8} :\ k = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$ and $x = \frac{\sqrt{3}}{2}$

$$I = \frac{1}{\sqrt{2}}B\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \sin\theta = \frac{\sqrt{2}}{4} \therefore \theta = 20^{\circ}42' \; and \; \sin\theta = \frac{\sqrt{3}}{2} \therefore \; \theta = \; 60^{\circ}$$

From tables,

$\mathrm{B}\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right) = 1.029$

$$\Longrightarrow I = \frac{1}{\sqrt{2}}\mathrm{B}\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{2}}(1.029) = 0.0728$$

Questions

Evaluate the following

1. $I \;= \int_o^3 \frac{x^3 dx}{\sqrt{3-x}}$
2. $I \;= \int_o^1 x^4\,(1-x^2)^{\frac{1}{2}}\,dx$
3. $I \;= \int_o^{1/2} \frac{du}{5-6u-u^2}$
4. $I \;= \int_o^{\infty} x^4\, e^{-3x} dx$
5. Express the following in elliptic function
   i. $\int_o^{1/2} \sqrt{1+4\sin 2\theta}\, d\theta$
   ii. $\int_0^1 \sqrt{\frac{4-k^2}{1-x^2}}\, dx$
   iii. $\int_{0.5}^1 \frac{dx}{\sqrt{3-4x^2+x^4}}$

9.1.4 Special function from linear O.D.E

Special functions are solutions obtained from certain frequently occurring linear second order differential equations such as Legendre functions are the solutions obtained from Legendre differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \; n(n+1)y = 0 \qquad 9.22$$

Bessel functions are the functions obtained from Bessel's differential equation given as

$$x^2\frac{d^2y}{dx^2} + \; x\frac{dy}{dx} + (x - v^2)y = 0 \qquad 9.23$$

These functions have many representations: as specified solutions of given differential equation, series, various integral representations, by recurrence relations, and by generating functions. Starting from one of these representations one can obtain the others. Thus in this section the path followed to get from one to the other is not unique but rather illustrative hyper geometric functions which include many other functions as special case ar3e mentioned. Mathieu functions are just mentioned here in this text without treatment of further discussion.

9.1.5 Legendre function

Starting with the Legendre's equation, the solution given polynomial

$$P_n(x) = k_n \sum(-1)^r \frac{(2n-2r)!}{(n-r)!} \frac{x^{n-2r}}{(n-2r)!} \qquad 9.24$$

Where k_n is arbitrary and is fixed to be $\frac{1}{2^n}$ with this and some factorial manipulation equation becomes

$$\mathrm{P_n(x)} = \left(\frac{d}{dx}\right) \frac{d^n}{dx^n} (x^2 - 1)^n \qquad 9.25$$

Equation (4) is known as Rodriguez' formula for the Legendre polynomials

The first few for $n = 0, 1, 2, 3, 4$and 5. are

$$\mathrm{P}_0(x) = 1, \qquad \mathrm{P}_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\mathrm{P}_1(x) = \mathrm{x} \qquad \mathrm{P}_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$\mathrm{P}_2(x) = \frac{1}{2}(3x^2 - 1)\mathrm{P}_5(x) = \frac{1}{8}(35x^4 - 70x^3 + 15x)$$

A contour integral representation of Rodriguez' formula makes use of Cauchy's formula.

$$f(\mathrm{z}) = \frac{1}{2\pi i} \oint \frac{\mathrm{f(t)dt}}{\mathrm{t - z}} \qquad 9.23$$

Provided *f*(t) is regular within the contour, which encloses z once in a positive sense. Differentiating n times with respect to z gives

$$\left(\frac{\mathrm{d}}{\mathrm{dz}}\right)^{\mathrm{n}} f(\mathrm{z}) = \frac{\mathrm{n!}}{2\pi i} \oint \frac{f(t)dt}{(1 - z)^{n+1}} \qquad 9.24$$

Therefore, from equation (4) we obtain

$$\mathrm{P_n(z)} = \frac{1}{2^{\mathrm{n}}} \frac{1}{3\pi i} \oint \frac{(t^2 - 1)^{\mathrm{n}}}{(t - z)^{n+i}} dt \qquad 9.25$$

Legendre functions is an orthonormal basis for the space (c-1, 1) 0f all piecewise continuous and smooth real valued functions on the interval (-1, 1).

Let C(-1, 1) be the space of all piecewise continuous and smooth complex valued functions on the interval (-1, 1) and let A be the linear operator defined on C(-1, 1) by

$Ay = (1 - x^2)y^{11}(x) - 2xy(x)$

Or $Ay = \frac{d}{dx}\left[(1 - x^2)\frac{d}{dx}\right] y(x)$

$\forall\, y \in C(-1, 1)$ and $x \in (-1, 1)$

Then the eigenproblem of A given by

$$(1 - x^2)y^{11}(x) - 2xy^1(x) = -\lambda y(x), -1 < x < 1$$

is called Legendre's differential equation by Frobenuis method it follows that the Legendre's eigenproblem possesses eigenvalues λ given by

$$\lambda = \iota(\iota + 1)$$

and corresponding eigenfunctions given by $y_\iota(x) = p_\iota(x)$ which is known as the Legendre Polynomials and is expressed in a closed form as

$$P_\iota(x) = \frac{1}{2^\iota \iota!}\frac{d^\iota}{dx^\iota}(x^2 - 1)^\iota$$

This is called the Rodriguez' formula. The polynomial here can be normalized in a traditional way for $\iota = 0, 1, 2, 3, 4, 5$ etc.

They have simple symmetry properties such that

$$P_\iota(-x) = (-1)^\iota P_\iota(x)$$

The few first ones are

$$P_0(x) = 1,$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = (5x^3 - 3x^3)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x + x)$$

Example by Rodriguez' formula obtain

$$P_5(x)$$

Solution

$$P_\iota(x) = \frac{1}{2^\iota \iota!}\frac{d^\iota}{dx^\iota}\,(x^2 - 1)^\iota$$

When $\iota = 5$

$$P_5(x) = \frac{1}{2^5 5!}\frac{d_5}{dx^5}\,(x^2 - 1)^5$$

$$= \frac{1}{2^5 5!}\frac{d_5}{dx^5}(x^{10} - 5x^8 + 10x^6 - 10x^4 + 5x^2 - 1)$$

$$= \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Associated Legendre differential Equation and Function.

The generalization of Legendre differential equation given by

$$[\iota - x^2 y^{11}(x)]^1 + \left[\iota(\iota + 1) - \frac{m^2}{1 - x^2}\right]y(x) = 0,$$

$$-1 < x < 1$$

Where $\iota = 0, 1, 2 \ldots$ is called the associated Legendre differential equation by Frobenius method, it follows that the associated Legendre differential equation possesses, for each value of ι, the eigen values

$$m = -\iota, -\iota + 1, \ldots \quad 0, \iota, \ldots\ldots \ , \iota$$

and corresponding everywhere to regular eigenfunctions.

$$y(x) = P_\iota^m(x)$$

Where

$$P_\iota^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\iota^m(x)$$

whereP_ι is the legendre function of order ι. The functions $P_\iota^m(x)$ are called the associated Legendre function of order ιm.

Example:

Obtain the associated Legendre function for $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Solution

To obtain m, $m = -\iota + 1$

$$\Rightarrow as\iota = 3, \quad m = 2$$

Therefore using the appropriate equation, we have

$$P_3(x) = (1-x^2)\frac{d^2}{dx^2}\left[\frac{1}{2}(5x^3 - 3x)\right]$$

$$= 15(x - x^2)$$

For $P_3^1(x)$, it means that $\iota = 2$, then $m = -2 + 1 = 1$

$$\Rightarrow P_3^1(x) = (1-x^2)^{1/2}\frac{d}{dx}\left[\frac{1}{2}(5x^2 - 3x)\right]$$

$$= \frac{1}{2}(x - x^2)^{1/2}(15x^2 - 3)$$

9.1.6 Hermite functions

The eigenproblem defined by

$$y^{11}(x) - 2xy^{11}x(x) = 2xy(x)$$

Is called Hermite differential equation. The equation is

$$Ay(x) = e^{x^2}\frac{d}{dx}\left[e^{-x^2}\frac{d}{dx}\right]y(x)$$

For linear operator A defined on $C(-\infty,\infty)$ where C is a defined smooth complex value function on the interval$(-\infty,\infty)$.

The solution of Hermite differential equation gives eigenvalues given by

$$\lambda = 2n + 1; n = 0, 1, 2 \ \ldots$$

and corresponding eigenfunction given

$$y(x) = H_n(x)$$

which is called the Hermite polygnomials of order n.

This is given by

$$H_n(x) = \sum_{k=0}^{n/2} \frac{(-)^k n! \ 2^{n-2k}}{K! \, (n-2k)!} x^{n-2k}$$

It can also be written as

$$H_n(x) = (-1)^4 e^{x^n} \frac{d^n}{dx^n} e^{-x^2}$$

$$n = 0.1, 2, ..$$

Example

Obtain the Hermite Polynomial for $n = 4$

$$\text{ie} H_4(x) = (-1)^4 e^{x^2} \frac{d^4}{dx^4} e^{-x^2}$$

$$= 16x^4 - 48x^2 + 12$$

9.1.7 Laguerre Function

This is the eigenfunctions obtained from the associated Laguerre differential equation. Laguerre differential equation is obtained from

$$XY^{11} + (\iota - x)y + ny = 0$$

where $n = 0, 1, 2 \ldots$

The solutions, the Laguerre Polynomials $L_n(x)$ are given by a Radrigues formula

$$L_n(x)e^x \frac{d^n}{dx^n}(x^n e^{-x}); m = 0, 1, 2, \ldots$$

Example:

For order $n = 3$, obtain $L_n(x)$

$$L_3(x) = e^x \frac{d^3}{dx^3}(x^3 e^{-x}) = -x^3 + 9x^2 - 18x + 6$$

By Frohonius method, it follows that the associated Laguerre eigenproblem for each value of n possesses eigenvalues $\curlywedge$equal any real number, with corresponding eigenfunctions.

$$y(x) = L_n^{\curlywedge}(x)$$

where

$$L_n^{\curlywedge}(x) = \frac{e^x e^{-\curlywedge}}{n!} \frac{d^n}{dx^n}(e^{-x} e^{\curlywedge + n})$$

and

$$\curlywedge = n + 1$$

which are called the associated Laguerre function of order $n \curlywedge$.

Example:

By definition

$$L_2^3(x) = \frac{e^x x^{-3}}{2\iota} \frac{d^2}{dx^2}(e^{-x} x^5) = \frac{1}{2}x^3 - 5x^7 + 10x^6$$

$$\text{and} L_2^{1/2}(x) = \frac{e^x x^{-1/2}}{2!} \frac{d^2}{dx^2}(e^{-x} x^5) = \frac{1}{2}x^2 - \frac{5}{2}x + \frac{15}{8}$$

Modified Bessel Function

The Bessel function $J_n(Kr)$ and the Neumann $Nn(Kr)$ function oscillate at large distance, provided that K is real. When K is purely imaginary, it is convenient to combine them so as to have functions that grow or decay exponentially. These are the modified Bessel functions.

We define

$$I_v(x) = \iota^{-r} Jr(\iota x)$$

$$K_r(x) = \frac{x}{25\,mv\pi}[I_{-r}(x) - I_r(x)]$$

At short distance

$$\mathrm{I}_r(x) = \left(\frac{x}{2}\right)^r \frac{1}{(r+1)}$$

$$K_v(x) = \frac{1}{2}\sqrt{v}\left(\frac{x}{2}\right)^{-v}$$

When v becomes an integer, we take a limits and in particular

$$\mathrm{I}_o(x) = 1 + \frac{1}{4}x^2 + \cdots,$$

$$K_o(x) = -\left(I_n\frac{x}{2} + Y\right) + \cdots$$

The large X asymptotic behaviour is

$$I_v(x) = \frac{1}{\sqrt{2\pi x}}e^x, \qquad x \to \infty,$$

$$K_v(x) = \frac{\pi}{\sqrt{2x}}e^{-x}, \qquad x \to \infty$$

The factor of ι^{-r} in the definition of $I_v(x)$ is to make I_r real.

From the expression for $J_n(x)$ as an integral is

$$I_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{\iota n\theta}\, e^{x cos\theta}\, d\theta = \frac{1}{\pi}\int_o^{\bar{n}} \cos(n\theta) e^{x cos\theta}\, d\theta$$

for integer n. When n is not an integer, the expression for $I_r(x)$ as an integral becomes

$$I_r(x) = \int_0^{\pi} \cos(r\theta) e^{x\cos\theta} d\theta - \frac{\sin r\pi}{\pi} \int_o^{\infty} -xcosht - vtdt$$

9.1.8Spherical Bessel Function

Consider the wave equation

$$\left(\nabla^2 - \frac{1}{C^2}\frac{\partial^2}{\partial t^2}\right)\varphi(r,\theta,\phi,t) = 0$$

In spherical polar coordinates.

To apply separation of variables, we get

$$\varphi = e^{iwt} Y_m^l(\theta,\phi)x(r).$$

and find that

$$\frac{d^2x}{dr^2} + \frac{2dx}{rdr} - \frac{\iota(\iota+1)}{r^2}x + \frac{w^2}{c^2}x = 0$$

Substitute$x = r^{-1/2}R(r)$and we obtain

$$\frac{d^2}{dr^2}x + \frac{1}{r}\frac{dR}{dr} + \left(\frac{w^2}{c^2} - \left(\frac{\iota+\frac{1}{2}}{r^2}\right)\right)R = 0$$

This is Bessel's equation with $r \to \left(\iota + \frac{1}{2}\right)^2$

whose general solution is

$$R = AJ_{\iota+\frac{1}{2}}(K_r) + BJ_{-\iota-\frac{1}{2}}(K_r)$$

where$K = \frac{w}{e}$

By inspection of the series definition of the J_r reveals that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

are Bessel functions where are actually elementary function. This is true of all Bessel functions of half integer order,

$$v = \neq {}^{1}/_{2}, \pm {}^{3}/_{2}, \ldots$$

We define the spherical Bessel functions by

$$j_{\iota}(x) = \sqrt{\frac{\pi}{2x}} J_{\iota+\frac{1}{2}}(x)$$

$$n_{\iota}(x) = (-1)^{1+1}\sqrt{\frac{\pi}{2x}} J_{-\left(\iota+\frac{1}{2}\right)}(x) - Schiff\ 19 \ldots$$

The first few are

$$j_0(x) = \frac{1}{x}\ \sin x$$

$$j_1(x) = \frac{1}{x^2}\sin x - \frac{1}{x}\cos x$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x$$

$$n_0(x) = -\frac{1}{x}\cos x$$

$$n_1(x) = \ -\frac{1}{x^2}\cos x - \frac{1}{x}\sin x$$

$$n_2(x) = \ -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - {}^{3}/_{x^2}\sin x$$

9.1.9 Bessel functions

As already mentioned before these are obtained from Bessel equation.

We defined the Bessel function

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\mathrm{r!\,(n+r+)!}} \left(\frac{x}{2}\right)^{\mathrm{n+2r}} \quad 9.26$$

From the recursion relation of the power series solution,

$$J_n -_1(x) J_{n+}(x) = n \sum_{r=0}^{\infty} \frac{(-1)^r}{\mathrm{r!\,\Gamma(m+r+1)}} J_{n-1}(x) - J_{n+1} \quad 9.27$$

$$\Rightarrow J_{n-1}(x) + J_{n+1}(x) = \frac{\mathrm{2n}}{\mathrm{x}} J_n(x) \quad 9.28$$

Similarly,

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n(x) \quad 9.29$$

Adding and subtracting equations

(11) And 12 give

$$J_{n-1}(x) = \frac{\mathrm{n}}{\mathrm{x}} J_n(x) + J1_n(x) \quad (9.30)$$

$$\frac{\mathrm{n}}{\mathrm{x}} J_n(x) J_n(x) \quad 9.31$$

Bessel functions of half-integral order may be simply expressed in terms of trigonometric functions and are

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} sinx \quad 9.32$$

$$J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} cosx \quad 9.34$$

From the recursion relations equations (13) and (14)

$$J_{\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \left(\frac{1}{x}\sin x - \cos x\right) \quad 9.35$$

$$J_{-\frac{3}{2}}\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left(-\frac{1}{x}\cos x - \sin x\right) \qquad 9.36$$

There is an integral relation in Bessel function such as

$$\frac{dJ_0(x)}{dx}J_1(x) \qquad 9.37$$

$$-J_0(x) = \int J_1(x)dx \qquad 9.38$$

9.2.0 DIRAC DELTA FUNCTIONS

The Dirac delta functions are generalized functions which are point functions thus are not differentiable. A generalized function, which used often in some problems, is the **Step** function or the **Heaviside**, function defined as:

$$\begin{aligned} H(x-a) &= 0 & x < a \\ &= {}^{1}/_{2} & x = a \\ &= 1 & x > a \end{aligned}$$

which is not differentiable at x=a. One should note that:

$$H(x-a) + H(a-x) = 1$$

9.2.1Definitions and Integrals

The one-demensional **Dirac delta** function $\delta(x-a)$ is one that is defined only through its integral. It is a point characterized by the following properties:

Definition:

$$\begin{aligned} \delta(x-c) &= 0 & x \neq c \\ &= \infty & x = c \end{aligned}$$

Integral:

Its integral is defined as:

$$\int_{-\infty}^{\infty} \delta(x-c)dx = 1$$

Shifting Property:

Given a function$f(x)$, which is continuous at $x = c$, then:

$$\int_{-\infty}^{\infty} f(x)\delta(x-c)dx = f(c)$$

Shift Property:

This property allows for a shift of the point of application of $\delta(x-c)$, i.e:

$$\int_{-\infty}^{\infty} \delta(x-c)f(x)dx = \int_{-\infty}^{\infty} \delta(x)f(x+c)dx = f(c)$$

Scaling Property:

This property allows for the stretching of the variable x:

$$\int_{-\infty}^{\infty} \delta\left(\frac{x}{a}\right) f(x)dx = |a|f(0)$$

and

$$\int_{-\infty}^{\infty} \delta(x-c)/a)f(x)dx = |a|f(c)$$

Even Function:

The Dirac function is an even function, i.e:

$$\delta(c-x) = \delta(x-c)$$

since:

$$\int_{-\infty}^{\infty} \delta(c-x)dx = 1$$

and

$$\int_{-\infty}^{\infty} \delta(x-c)f(x)dx = f(c) = \int_{-\infty}^{\infty} \delta(c-x)f(x)dx$$

9.2.2Definite Integrals:

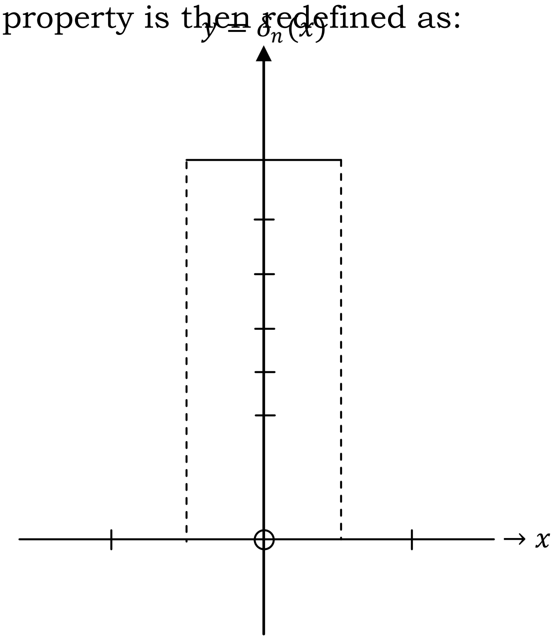
The Dirac delta function may be integrated over finite limits, such that:

$$\int_a^b \delta(x-c)dx = 0 \qquad c < a, or\ c > b$$

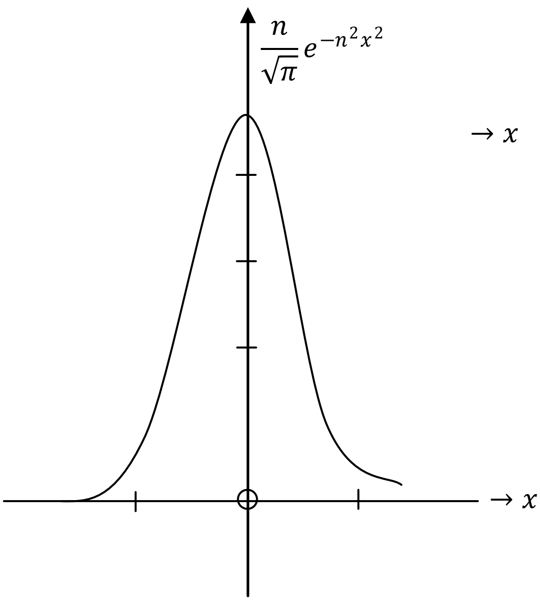
$$= 1/2 \qquad c = a, or\ c = b$$

$$= 1 \qquad a < c < b$$

and the sifting property is then redefined as:



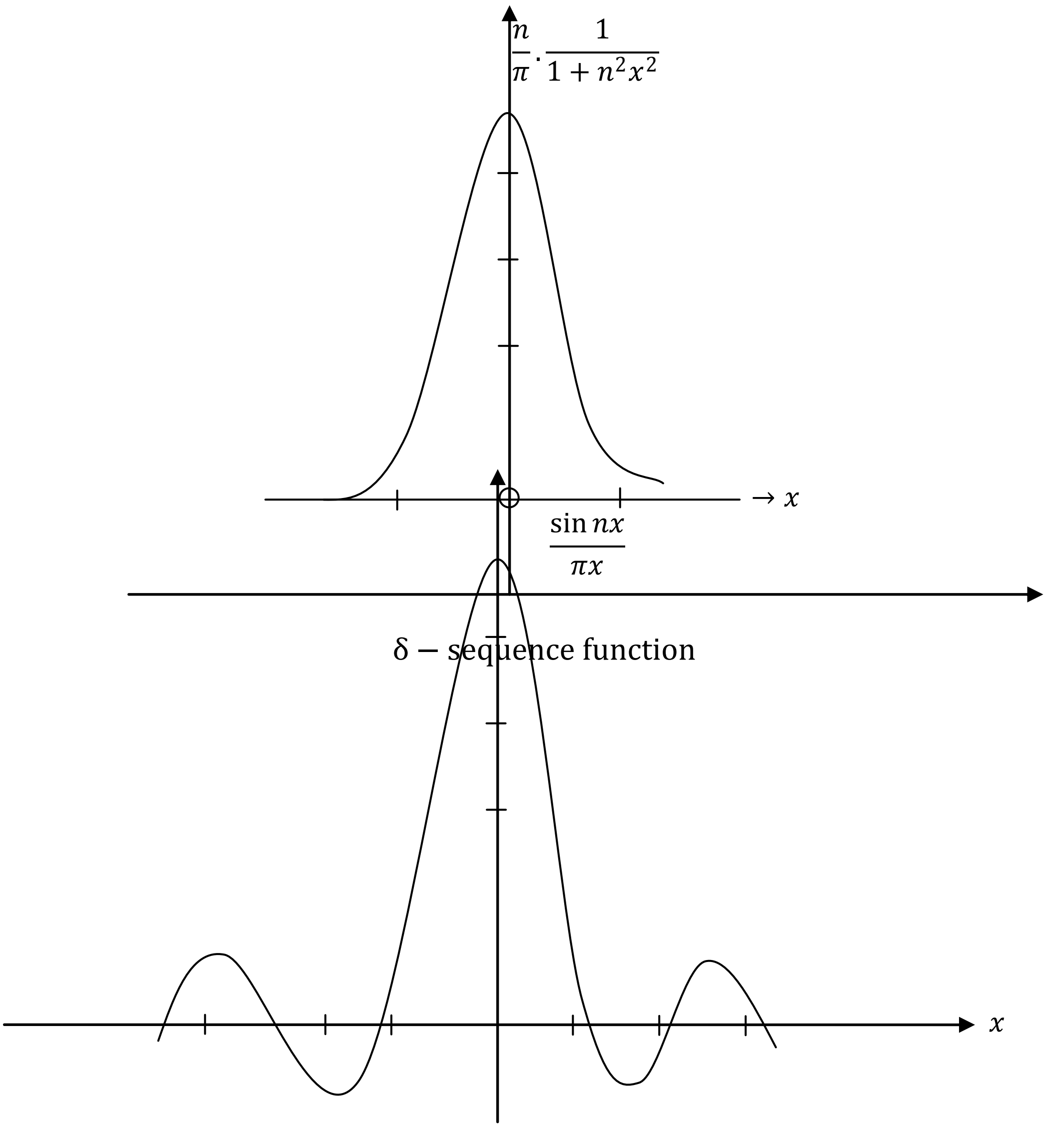
δ − sequence function



δ − sequence function

These approximations have varying degrees of usefulness. Equation is useful in providing a simple derivation of the integral property. the equation ois convenient to differentiate. Its derivatives lead to the Hermite Polynomials. Equation is particulary useful in Fourier analysis nad in its application to quantum mechanics. In the theory of Fourier series, often appears (modified) as the Dirichlet Kernel:

$$\delta_n(x) = \frac{1}{2\pi} \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{1}{2}x\right)}$$



In using these approximations considering well behaved system, we assume that $f(x)$ is well behaved-it offers no problems at large x.

For most physical purposes such approximations are quite adequate. However, from a mathematical point of view the situation is still unsatisfactory: The limits

$$\lim_{n\to\infty} \delta_n(x)$$

Do not exist.

$$\int_a^b f(x)\delta(x-c)dx = 0 \qquad c < a, or\ c > b$$

$$= 1/2f(c) \qquad c = a, or\ c = b$$

$$= f(c) \qquad a < c < b$$

If the integral is an indefinite integral, the integral of the Dirac delta function is a Heaviside function:

$$\int_{-\infty}^{x} \delta(x-c)dx = H(x-c)$$

and

$$\int_{-\infty}^{x} \delta(x-c)\, f(x)dx = f(c)H(x-c)$$

9.1.3 Integral Representation

One can define continuous, differentiable functions which behave as a Dirac delta function when certain parameters vanish, i.e. let:

$$\lim_{a\to 0} u(a,x) - \delta(x)$$

If it satisfies the integral and shifting properties above.

To construct such representations, one may start with improper integrals whose values are unity, i.e. let $U(x)$ be a continuous even function whose integral is:

$$\int_{-\infty}^{\infty} U(x)dx = 1$$

then a function representation of the Dirac delta function when $a \to 0$ is:

$$u(a,x) = U(x/a)/a$$

which also satisfies the shifting property in the limit as $a \to 0$

Example

The function $u(a,x) = a/[\pi(x^2 + a^2)]$ behaves like $\delta(x)$, since:

$$\lim_{a\to 0} u(a,x) \to \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

and since it satisfies the integral and shifting properties:

$$\int_{-\infty}^{x} u(a,x)dx = \frac{1}{2} + \frac{1}{\pi}\arctan(\frac{x}{a})$$

Dirac delta function can be used to overcome difficulties in evaluating integrals in a closed form.

Dirac Delta Function of Order One

The Dirac delta function of order one is defined formally by

$$\delta_1(x - x_0) = -\frac{d}{dx}\delta(x - x_0)$$

such that its integral vanishes:

$$\int_{-\infty}^{\infty} \delta_1(x - x_0)dx = 0$$

and its first moment integral is unity:

$$\int_{-\infty}^{\infty} x\delta_1(x - x_0)dx = 1$$

and its sifting property is given by:

$$\int_{-\infty}^{\infty} f(x)\delta_1(x - x_0)dx = f`(x_0)$$

which gives the value of the derivative of the function $f(x)$ at the point of application of $\delta_1(x) = -\underset{a\to 0}{\text{Lim}}\frac{d\,u(a,x)}{d\alpha}$

In physical applications, $\delta_1(x)$ represents a mechanical concentrated couple or a dipole.

9.1.4 Dirac Delta Function of Order N

These Dirac delta function of order N can be formally defined as:

$$\delta_N(x - x_0) = (-1)^N \frac{d^N}{dx^N}\delta(x - x_0)$$

so that the K^{th} moment integral is:

$$\int_{-\infty}^{\infty} x^K \delta_N(x)dx = \begin{cases} 0 & K < N \\ N! & K = N \end{cases}$$

so that when the upper limit infinite, the integral approaches unity. Note also that if the limit of the integral is taken when $\alpha \to 0$, the integral approaches $H(x)$. It should be noted that this functional representation was obtained from the integral:

so that:

$$U(x) = \frac{1}{\pi(1+x^2)}$$

which results in the form given for $u(\alpha, x)$ above. To satisfy the sifting property, one may use a shortcut procedure which assumes uniform convergence of the integrals, i.e:

$$\lim_{\alpha \to 0} \int_{-\infty}^{\infty} u(\alpha, x) f(x)dx = \frac{1}{\pi} \lim_{a \to 0} \int_{-\infty}^{\infty} \frac{a}{x^2 + a^2} f(x)dx$$

Substituting $y = x/a$ in the above integral one obtains:

$$\lim_{\alpha \to 0} \int_{-\infty}^{\infty} u(\alpha, x) f(x)dx = \frac{1}{\pi} \lim_{a \to 0} \int_{-\infty}^{\infty} \frac{f(ay)}{1+y^2} dy \to \frac{f(0)}{\pi} \int_{-\infty}^{\infty} \frac{dy}{1+y^2} = (0)$$

Where the integral is assumed to be uniformly convergent in α. Let $f(x)$ be absolutely integrable and continuous at $x = 0$, then one can perform these integrations without this assumption by integration by parts:

$$\frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{x^2+\alpha^2} dx = \frac{\alpha}{\pi} \left\{ \int_{-\infty}^{0} \frac{f(x)}{x^2+\alpha^2} dx + \int_{0}^{\infty} \frac{f(x)}{x^2+\alpha^2} dx \right\}$$

$$= \frac{\alpha}{\pi} \left\{ \int_{0}^{\infty} \frac{f(-x)}{x^2+\alpha^2} dx + \int_{0}^{\infty} \frac{f(x)}{x^2+\alpha^2} dx \right\}$$

Integrating the second integral by parts:

$$\lim_{\alpha \to 0} \left(\frac{\alpha}{\pi} \int_{0}^{\infty} \frac{f(-x)}{x^2+\alpha^2} dx \right) = \lim_{\alpha \to 0} \left(\frac{1}{\pi} f(x) arctan\,(x/\alpha)|_0^\infty - \frac{1}{\pi} \int_{0}^{\infty} f`(x)dx \right)$$

$$= -\frac{1}{\pi} \lim_{\alpha \to 0} f(x) \arctan(x/\alpha)\, dx \to -\frac{1}{2} \int_{0}^{\infty} f`(x)dx$$

$$= -\frac{1}{2} f(x)|_0^{\infty} = \frac{1}{2} f(0^+)$$

Since $f(x)$ is absolutely integrable and continuous at $x = 0$. Similarly the first integral

$$\lim_{\alpha \to 0} \left(\frac{\alpha}{\pi} \int_0^{\infty} \frac{f(-x)}{x^2 + \alpha^2} dx \right) \to \frac{1}{2} f(0^-)$$

so that, since $f(x)$ is continuous at $x = 0$:

$$\lim_{\alpha \to 0} \left(\int_{-\infty}^{\infty} f(x) u(\alpha, x) dx \right) \to f(0)$$

9.1.5 Transformation Property

One can represent a finite number of Dirac delta functions by one whose argument is a function. Consider o$\delta[f(x)]$ where $f(x)$ has a non-repeated null at x_0 and whose derivative does not vanish at x_0, then one can show that:

$$\delta[f(x)] = \frac{\delta(x - x_0)}{|f`(x_0)|}$$

One can show that this is correct by satisfying the conditions on integrability and the sifting property. Starting with the integral of $\delta[f(x)]$:

$$\int_{-\infty}^{\infty} \delta[f(x)] dx$$

Letting:

$$u = f(x)$$

then:

$$u = 0 = f(x_0) \qquad and \qquad du = f`(x) dx$$

then the integral becomes:

$$\int_{-\infty}^{\infty} \delta[f(x)] dx = \frac{1}{|f`(x_0)|} = \frac{1}{|f`(x_0)|} \int_{-\infty}^{\infty} \delta(x - x_0) dx$$

and

$$\int_{-\infty}^{\infty} \delta[f(x)] F(x) dx = \int_{-\infty}^{\infty} \frac{\delta(u)}{|f`(x(u))|} F(x(u)) du$$

$$= \frac{F(x_0)}{|f`(x_0)|} = \frac{1}{|f`(x_0)|} \int_{-\infty}^{\infty} F(x)\delta(x - x_0)dx$$

Thus, the two properties are satisfied if $\frac{r(x-x_0)}{|f`(x_0)|}$ represents $\delta[f(x)]$.

If $f(x)$ has a finite or an infinite number of non-repeated zeroes, i.e:

$$f(x_n) = 0 \qquad n = 1, 2, 3, \dots N$$

then:

For instance in solving this equation the solution can be written ordinarily-4π or o

$$\int \nabla . \nabla \left(\frac{1}{r}\right) d\tau = -\int \nabla . \left(\frac{r}{r^2}\right) d\tau = \{0.^{-} 4\pi$$

Depending on whether or not the integration include the origin r =0. This result may be conveniently expressed by introducing the Dirac delta function

$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(r) = -4\pi\delta(x)\delta(y)\delta(z).$

This Dirac delta function is defined by its assigned properties

$$\delta(x) = 0, x \neq 0$$

$$f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx \qquad A_1$$

Where $f(x)$ is any well-behaved function and the integration includes the origin in a special case of the proceeding equation

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

From the same equation$\delta(x)$ must be an infinitely high, thin spike at $x = 0$, as in the description of an impulsive force or the charge density for a point charge. The problem is that no such function exists in the usual sense of function. However, the crucial property can be developed in the equation rigorously as the limit of a sequence of functions, a distribution. For example, the delta function may be approximated by the sequences of functions, Eqs. As written below

$$\delta_n(x) = \begin{cases} 0, x < -\frac{1}{2n} \\ n, -\frac{1}{2n} < x < \frac{1}{2n} \\ 0, x > \frac{1}{2n} \end{cases}$$

$\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp(-n^2x^2)$

$$\delta_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}$$

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} e^{ixt}\,\partial t.$$

Example D.3

The following integral, which is known to have an exact value, can be approximately evaluated for small values of its parameters c:

$$T = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(a\times)J_0}{X^2+C^2}d\times = \frac{1}{c}e^{-ac}1_0(bc) \cong \frac{1}{c} \qquad c << 1$$

If the integral cannot be evaluated in a closed form and one would like to evaluate this integral for small values of c, one notices that the function in example D.1,

C $/[\pi(\times^2 + c^2)]$, behaves as $\delta(\times)$ in the limit of $c \longrightarrow 0$. Thus, one can approximately evaluate the integral by the sifting properties. Letting:

$$F(\times) = \frac{1}{c}\cos(a\times)J_o(b\times)$$

Then the sifting property gives $F(0) = 1/c$. To check the numerical value of this approximation, one can evaluate it exactly, so that for a =b =1 one obtains:

C	T(exact)	T(approx)	cT(exact)	cT(approx)
0.2	4.13459	5.000	0.82692	1.0
0.1	9.07090	10.00	0.90709	1.0
0.01	99.0050	100.00	0.99005	1.0

This example shows that for c =0.1 the error is within10 percent of its exact value. This approximate method of evaluating integrals when parts of the integrand behave like

A way ort if this difficulty is provided by the theory of distributions. Recognizing that Eg. $f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)d1$ Is the fundamental property, we focus our attention of it rather than on$\delta(x)$itself. These other equations with n =1, 2, 3… may be interpreted as sequences of normalized functions:$\int_{-\infty}^{\infty}\delta_n(x)dx = 1.$

The sequence of integrals has the limit$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)f(x)dx = f(0).$

Not that this equation is the limit of a sequence of integrals. Again, the limit of $\delta_n(x), n \to \infty, does\ not\ exist.$ (The limits for all four forms of $\delta_n(x)$ diverge at $x = 0.$)

We may treat $\delta(x)$ consistently in the form $\int_{-\infty}^{\infty} \delta(x) f(x) dx = \lim_{n \to \infty} \int_{-\infty}^{\infty} \delta_n(x) dx$ C1$\delta(x), n \to \infty,$ does not exist. (The limits for all four forms of $\delta_n(x)$ as indicated in the above. We might emphasize that the integral in the left-hand side of the equation is not a Riemann integral, but rather a limit. This distribution $\delta(x)$ is only one of a infinite number of possible distributions, of the equation but it is the one we are interested in because $f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx$ From these sequences of functions we s that the Dirac delta functions must be even in $x, \delta(-x) = \delta(x)$.

The integral property,$f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx$, is useful in cases where the argument of the delta function is a function $g(x)$ with simple zeros on the real axis, which leads to the rules

$$\delta(ax) = \frac{1}{a}\delta(x), \qquad a > 0,$$

$$\delta(g(x)) = \sum_{\substack{a, \\ g(a)=0, \\ g'(a)\neq 0}} \frac{\delta(x-a)}{|g'(a)|}.$$

To obtain Eq. (1.179) we change the integration variable in $\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \frac{1}{a}\int_{-\infty}^{\infty} f(\frac{y}{a})\delta(y)dy = \frac{1}{a}f(0)$, and apply Eq. A1. To prove Eq. A2 w3 de3compose the integral $\int_{-\infty}^{\infty} f(x)\delta(g(x)0dx = \sum_a \int_{a-\varepsilon}^{a+\varepsilon} f(x)\delta 9(x-a)g'(a))dx$
A3

Into a sum of integrals over small intervals containing the zeros of $g(x)$. in these intervals,$g(x) \approx g(a) + (x-a)g'(a) = (x-a)(a)$. Using Eq.B1 A3 right-hand side of Eq. we obtain the integral of Eq. A2 delta function by the relation

$\int_{-\infty}^{\infty} f(x)g'(x-x')dx = -\int_{-\infty}^{\infty} f'(x)\delta(x-x')dx = -f'(x')$. We use $\delta(x)$ frequently and call it the Dirac delta function __ for historical reasons, nut remember that if is not really a function. It is essentially a shorthand notation, defined implicitly as the limit of integrals in a sequence $\delta_n(x)$ according to Eq. It should be understood that our Dirac delta function has significance only as part of an integrand. In this spirit the Dirac delta function is often regarded as an operator, a linear operator $\int \delta(x-xo)f(x)dx = f(xo)$ it may also be classified as a linear mapping or simply as a generalized function. Shifting our singularity to the point$x = x'$, we write the Dirac delta function as $\delta(x-x')$. A1 becomes

$\int_{-\infty}^{\infty} f(x)\delta(x-x^{'})dx = f(x^{'})$. As a description of a singularity at $x = x^{'}$, the Dirac delta function may be written as $\delta(x-x')$ or as $\delta(x^{'}-x)$. going to three dimension and using spherical polar coordinates, we obtain $\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\delta(r)r^2 dr \sin\theta\, d\theta d\varphi = \iiint_{-\infty}^{\infty}\delta(x)\delta(y)\delta(z)dx\,dy\,dz =$ d1

This corresponds to a singularity (or source) at the origin. Again, if our source is at$r = r_1$,Eq. d1 becomes $\iiint \delta(r_2 - r_1) r_{2^2 dr_2 \sin\theta_2\, d\theta_2\;\; d\varphi_2 = 1}$

9.1.6 Delta Function Representation by Orthogonal Functions

Dirac's delta function can be expanded in terms of any basis of real orthogonal functions $\{\varphi_n(x), n = 0,1,2,\}$ in solutions of ordinary differential equations of the Sturm-Liouville form. The satisfy the orthogonility relations

$$\int_a^b \varphi m(x)\varphi_n(x)dx = \delta_{mn} c_2$$

$$\int_a^b \varphi_{m(x)}\varphi_n\,(x)dx = \delta mn$$

Where the interval (a,b) may be infinite at either end or both {for convenience we assume that φ_n has been defined to include $\left(w(x)\right)^{1/2}$ if the orthogonality relations contain an additional positive weight function $\omega(x)$.} we use the φ_n to expand the delta function as

$\delta(x-t) = \sum_{n=0}^{\infty} a_n(t)\varphi_n(x),$

Where the coefficients a_n are functions of the variable t. Multiplying by $\varphi_m(x)$ and integrating over the orthogonality interval

$$\int_a^b \varphi_m(x)\,\varphi_n(x)dx = \delta_{mn}$$

$$\int_a^b \varphi_m(x)\varphi_n(x)dx = \delta_{mn,}(c_2)$$

We have $a_{m(t)} = \int_a^b \delta\,(x-t)\varphi_m(x)dx = \varphi_m(t)c_3$

Or $\delta(x-t) = \sum_{n=0}^{\infty}\varphi_n\,(t)\varphi_n(x) = \delta(t-x).$

This series is assuredly not uniformly convergent but it may be used as part of an integrand in which the ensuing integration will make it convergent suppose we form the integral $\int F(t)\,\delta(t-x)dx$, where it is assumed that $F(t)$ can be expanded in a series of orthogonal functions $\varphi_p(t)$, a property called completeness. We then obtain$\int F\,(t)\delta(t-x)dt = \int \sum_{p=0}^{\infty} a_p\varphi_p(t)\sum_{n=0}^{\infty}\varphi_n\,(x)\varphi_n(t)dt$

$$= \sum_{p=0}^{\infty} a_p \varphi_p(x) = F(x),$$

The cross products $\int \varphi_p \varphi_n \, dt (n \neq p)$ vanishing by orthogonality Referring back to the definition of the Dirac delta function, we see that our series representing, eq.c_3 Satisfies the defining property of the Dirac delta function and therefore is a representation of it, this representation of the Dirac delta function is called Closure. The assumption of completeness of a set of functions for expansion of $\delta(x-t)$ yields the closure relation. The converse that closure implies completeness use the topic of

9.1.7 Integral Representation for the Delta Function

Integral transforms such as the Fourier integral

$$F(\omega) = \int_{-\infty}^{\infty} f\,(t) exp(i\omega t) dt$$

Lead to the corresponding integral representations of Dirac's delta function. For example, take

$$\delta_n(t-x) = \frac{\sin n(t-x)}{\pi(t-x)} = \frac{1}{2\pi} \int_{-n}^{n} exp\big(i\omega(t-x)\big) dw, c_3$$

Using Eq.(1.174). we have

$$f(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t) \delta_n(t-x) dt, c_4$$

Where $\delta_n(t-x)$ is the sequence in Eq.(c_1) defining the distribution $\delta\,(t-x)$.

Note that Eq.(c_4) Assumes that $f(t)$ is continuous at $t = x$. If we substitute Eq. (c_3)into Eq. (c_4)we obtain

$$f(x) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \int_{-n}^{n} exp\,\big(i\omega(t-x)\big) d\omega \, dt.$$

Interchanging the order of integration and then taking the limit as $n \longrightarrow \infty$, we have the Fourier integral theorem with the understanding that it belongs under and integral sign as in Eq.(c_4) the identification

$$\delta(t-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} expi\big(i\omega(t-x)\big) d\omega$$

Provides a very useful integral representation of the delta function when the Laplace transform

$$L\delta(s) = \int_0^{\infty} exp\,(-st)\delta(t-t_0) = exp(-st_0), \qquad t_0 > 0$$

Is inverted, we obtain the complex representation

$\delta(t-t_0) = \frac{1}{2\pi i}\int_{Y-i\infty}^{Y+i\infty} exp\,(s(t-t_0))ds,$

This is essentially equivalent to the previous Fourier representation of Dirac's delta function.

$$\delta[f(x)] = \sum_{n=1}^{N} \frac{\delta(x-x_n)}{|f`(x_n)|}$$

Example

$$\delta(x^2-a^2) = \frac{1}{2a}[\delta(x-a)+\delta(x+a)]\delta[cosx] = \sum_{n=-\infty}^{\infty} \delta\left[x-\frac{2n+1}{2}\pi\right]$$

9.1.8 Concentrated Field Representation

The Dirac delta function is often used to represent concentrated fields such as concentrated forces and monopoles. For $a\,X_0$ of magnitude P_0 can be represented by $P_0\delta(x-x_0)$. This property can be utilized in integrals of distributed fields where one component of the integrand behaves like a Dirac delta function when a parameter in the integrand is takes to some limit.

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