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On Survey of Constraint Qualifications for Nonlinear Optimization Problems

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ABSTRACT

In this paper, we survey the constraint qualifications for nonlinear optimization problems. Although constrained nonlinear problems involve the Karush-Kuhn-Tucker (KKT) and associated optimality conditions, some problems do not satisfy the KKT conditions at a minimum point. We discuss some assumptions on constraint set which guarantees the KKT conditions to hold at a minimum point. We approach our study on constraint qualifications and their relationships with the concept of some cones and their polar.

Keywords: Optimality Conditions, KKT theorem, Cones and Constraint Qualifications

INTRODUCTION

Optimization Problems, which seek to minimize or maximize real valued functions, play important roles in the real world. They can be linear or nonlinear optimization problems. Many applications in engineering, management science and operations research can be formulated as nonlinear constrained optimization problems and nonlinear optimization problems are further classified into unconstrained and constrained optimization problems, [1].

[2] traced the history of optimization to an ancient princess named Dido which fled from the prosecution of her brother and a piece of land on the Mediterranean coast caught her fancy. She made a deal with the local leader, requesting him to cut a bull's hide into thin strips and tie them up to enclose as much land as one can with it. This problem in a modern day language later became a mathematical Isoperimetric problem in Calculus of Variations which seeks to find the length that can enclose the maximum area among all closed curves of given lengths. While [3] defined optimization as the action of finding the solution (product mix, allocation of resources, investments, etc.) that leads to the best result - the highest profit, or output, or return, or the one that achieves the lowest cost, or waste. Its model consists of an objective function and a set of constraints expressed in the form of a system of equations or inequalities. The objective function is a measure of effectiveness, often the cost or the profit. The model also includes decision variables and parameters.

Generally, the existence of solutions for nonlinear optimization problems are guaranteed by the Existence Theorems but the optimal solutions for both the unconstrained and the constrained nonlinear optimization problems are verified by their first order necessary optimality conditions, their second order necessary and their second order sufficient optimality conditions. Although nonlinear optimization problems are classified into unconstrained and constrained optimization problems, only few problems can actually be formulated as unconstrained optimization problems in practice [4] but the conditions for the unconstrained nonlinear optimization problems are familiar to students and easy to solve.

Methods for solving constrained nonlinear optimization problems are grouped into these two approaches: transformational which convert constrained nonlinear optimization problems into another form before solving it -well-known methods include penalty methods, barrier methods, Lagrangian methods, and sequential quadratic programming methods. Non-transformational approach, work on original problems directly by searching through its feasible region for the optimal solutions, [5].There are two types of constrained nonlinear optimization problems: those subject to equality constraints and those subject to inequality constraints. Optimality conditions for equality constrained optimization problems involve the Lagrangian and associated optimality conditions. The solution of problems with inequality constraints and/or variable sign restrictions relies on Kuhn-Tucker theory, [6] However, some constrained nonlinear optimization problems do not satisfy the Karush-Kuhn-Tucker (KKT) conditions at the minimum point. The concept of constraints qualifications therefore examines the conditions that constraints must satisfy in order to ensure that the minimum point satisfies the (KKT) conditions **[7].** This condition obviously, guarantees the existence of Lagrange's multipliers which transform the constrained nonlinear optimization problems into unconstrained nonlinear optimization problems.

[8] did a study on some Mathematical Programs with Vanishing Constraints which had a number of important applications in structural and topology optimization, but typically does not satisfy standard constraint qualifications like the linear independence and the Mangasarian-Fromovitz constraint qualification. The result therefore stated that the Abadie constraint qualification is also typically not satisfied, whereas the Guignard constraint qualification holds under fairly mild assumptions for a particular class of optimization problems. [9] studied the Constant-Rank Condition and Second-Order Constraint Qualification. It adopted the condition for feasible points of nonlinear programming problems as was defined by and also referred to its proof that the constant-rank condition is a first-order constraint qualification. [10] also proved that the constant-rank condition is also a second-order constraint qualification and defined other second-order constraint qualifications. It concluded that constant-rank constraint qualification seems to be a useful tool for the analysis of convergence of some nonlinear programming methods, proved that CRCQ is in fact a strong second-order constraint qualification and also proved that, under this constraint qualification, a minimum point verifies the strong second-order necessary condition for any KKT multiplier.

In this paper, we investigate five (5) constraint qualifications and the relationships between them, and approach the Guignard and Quasi-regularity Constraint Qualifications with the concept of some important cones, Linear Independence and Mangasarian-Fromovitz's Constraint Qualifications with the concept of the regularity of x^* and the Slater's Constraint Qualifications by the concept of the linearity of h and the convexity of g and show that the Guignard Constraint Qualification (GCQ) is the weakest constraint qualification while the Linear Independence Constraint Qualification is the strongest constraint qualification.

PRELIMINARIES

Consider the nonlinear constrained optimization problem

$Min f(x)$,

$$Subject\ to \quad (1)\quad h(x) = 0,\ \ g(x) \leq 0.$$

Where $x \in \mathbb{R}^n$, and f: $\mathbb{R}^n \to \mathbb{R}, h: \mathbb{R}^n \to \mathbb{R}^m, g: \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable.The feasible set $\Omega = \{x \in \mathbb{R}^n: h(x) = 0, g(x) \leq 0\}$. Given that $x^* \in \Omega$, the classical Karush-Kuhn-Tucker (KKT) conditions

$$-\nabla f(x^*) = \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla g_j(x^*)$$

$${\mu^*}_j \geq 0, \quad j = 1, \dots, p,$$

$${\mu^*}_j g_j(x^*) = 0, \quad j = 1, \dots, p.$$

Although Karush-Kuhn-Tucker (KKT) conditions are necessary conditions for a given point to be the solution of a given constrained nonlinear optimization problem, some problems do not satisfy the (KKT) conditions at the minimum point.

We state the theorem of KKT

Theorem 1 (Karush-Kuhn-Tucker (KKT) Theorem):
Let $x^* \in \Omega$ be a local minimum of $f: \mathbb{R}^n \to \mathbb{R}$, subject to $h(x) = 0$, $g(x) \leq 0$, $h: \mathbb{R}^n \to \mathbb{R}^m$, g: $\mathbb{R}^n \to \mathbb{R}^p$ and m, $p \leq n$. Assume that x^* is a regular point. Then, there exist $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$
such that

$$\mu^* \geq 0$$

$$\nabla f(x^*)^t + \lambda^* \nabla h(x^*)^t + \mu^* \nabla g(x^*)^t = 0$$

$${\mu^*}^t g(x^*) = 0$$

where $\lambda_i s$ and $\mu_j s$ are called Lagrange multipliers and KKT multipliers respectively.

See Chong and Zak [3] for the proof
Although Karush-Kuhn-Tucker (KKT) conditions is very important in solving nonlinear inequality constraint problems,some problems do not satisfy the (KKT) conditions at the minimum point.

Example 1: Consider the nonlinear constrained optimization problem

$$Min f(x) \qquad (3)$$

Subject to

$$x_2 - (1 - x_1)^3 \leq 0$$

$$-x_2 \leq 0.$$

We note that the solution of problem (3) is $x^* = (1,0)^t$ but the KKT conditions are not satisfied at that point.

The problem of our study therefore is to investigate some assumptions on the constraints sets which guarantee the KKT conditions to be satisfied at the minimum point. Such assumptions are called constraint qualifications.

Let $x^* \in \Omega$, the active set of the inequality constraint $g_j(x) \leq 0$ at $x^* \in \Omega$ is the set

$$\mathcal{A}(x^*) = \{j: g_j(x^*) = 0\}. \qquad (4)$$

Equality constraints $h_i(x) = 0$ are always considered to be active.

Definition 1 (Convex Set): A set $C \subseteq \mathbb{R}^n$ is a convex set if for every

$x, y \in C$, $\alpha x + 1 - \alpha \in C$, $\forall \alpha > 0$

Definition 2 (Cone): A set $C \subseteq \mathbb{R}^n$ is a cone if for all $x \in C$, $\alpha x \in C$ for any $\alpha > 0$.

Definition 3 (Polar of a set): Given a set $S \subseteq \mathbb{R}^n$, the polar of S denoted by

$$S^o = \{p \in \mathbb{R}^n: p^T x \leq 0 \ \ \forall x \in S\}$$

We should note that for any $S \subseteq \mathbb{R}^n$, S^o is a cone and $S \subseteq (S^o)^o$ and the equality holds if S is a closed convex cone as stated below.

Lemma 1 (Farkas' Lemma): Let$C \subseteq \mathbb{R}^n$ be a closed convex cone. Then, $(C^o)^o = C$.

Proof;

Let, $x \in C$, we have $x^T y \leq 0, \forall y \in C^o \Longrightarrow x \in (C^o)^o$Let $z \in (C^o)^o \Longrightarrow C \subseteq (C^o)^o$ Conversely, Let $z \in (C^o)^o$and $\hat{z} = proj_C(z) \in C \Longrightarrow (z-\hat{z})^T(x-\hat{z}) \leq 0, \forall x \in C$. Taking $x = 0$ and $x = 2\hat{z}$, we obtain $(z-\hat{z})^T\hat{z} = 0 \Longrightarrow (z-\hat{z})^T x \leq 0 \forall x \in C \Longrightarrow (z-\hat{z}) \in C^o$ and since $z \in (C^o)^o$, we have $(z-\hat{z})^T\hat{z} = 0 \Longrightarrow \| z-\hat{z} \|^2 = 0 \Longrightarrow z = \hat{z}$ and $z \in C \Longrightarrow (C^o)^o \subseteq C = (C^o)^o$.

Definition 4 (Feasible direction): Let$x^* \in \Omega$, $d \in \mathbb{R}^n$ is a feasible direction at x^* with respect to Ω if $\exists \delta > 0$ such that, $x^* + \lambda d \in \Omega \forall \lambda \in [0,\delta]$.

Definition 5 (Cone of feasible direction): The cone of feasible direction of at x^* is the set $V(x^*) = \{d \in \mathbb{R}^n : x^* + \lambda d \in \Omega, \forall \lambda \in [0,\delta]$ for some $\delta > 0$.

Definition 6 (Cone of descent direction): Given a function f: $\mathbb{R}^n \to \mathbb{R}$, the cone of descent directionat x^*defined by$D(x^*) = d \in \mathbb{R}^n : f(x^* + \lambda d) < f(x^*)$, $\forall \lambda \in [0,\delta]$ for some $\delta > 0$.

Lemma 2: Let f:$\mathbb{R}^n \to \mathbb{R}$ be a differentiable function at $x^* \in \mathbb{R}^n$. Then,

$(i) \nabla f(x^*)^t d \leq 0$, for all $d \in D(x^*)$.
(ii) If $d \in \mathbb{R}^n$ satisfies $\nabla f(x^*)^t d < 0$, then $d \in D(x^*)$. We denote $D_0(x^*) = \{d \in \mathbb{R}^n : \nabla f(x^*)^t d < 0\}$
(5)

Proof: See Endris [5]

Definition 7 (Feasible Sequence): Let$x^* \in \Omega$, $\{z_k\}$ is called a feasible sequence if $z_k \to x^*, and z_k \in \Omega$ for k very large.**Definition 8 (Linearized Cone):** Let$x^* \in \Omega$, the analytic tangent cone is defined by

$$\mathcal{L}(x^*) = \{d \in \mathbb{R}^n : d^t \nabla h_i(x^*) = 0, \forall i = 1, \ldots, m,$$
$$d^t \nabla g_j(x^*) \leq 0 \ \ j \in \mathcal{A}(x^*)\}$$
(6)

Definition 9: Let $x^* \in \Omega$, the cone

$$G(x^*) = \{\textstyle\sum_{i=1}^m \lambda_i \nabla h_i(x^*) + \sum_{j \in \mathcal{A}(x^*)} \mu_j \nabla g_j(x^*) : \mu_j \geq 0, \forall j \in \mathcal{A}(x^*)\} \quad (7)$$

Lemma 3: For any $x^* \in \Omega$, $G(x^*)$ is a closed convex cone

Proof: See Endris [5]

Lemma 4: For any $x^* \in \Omega$, $G(x^*) = (\mathcal{L}(x^*))^0$

Proof: From Lemma 1 and Lemma 3, we prove that $G(x^*) = (\mathcal{L}(x^*))^0$.
Consider $d \in \mathcal{L}(x^*)$ and given, $v \in G(x^*)$, we have

$$d^t v = \{\textstyle\sum_{i=1}^m \lambda_i d^t \nabla h_i(x^*) + \sum_{j \in \mathcal{A}(x^*)} \mu_j d^t \nabla g_j(x^*). \quad (8)$$

Since $\mu_j \geq 0$ and by the definition, $\mathcal{L}(x^*)$, $d^t v \leq 0$. Hence, $d \in (G(x^*))^0$.

Conversely, consider, $d \in (G(x^*))^0$. By definition, $d^t v \leq 0$, $\forall\, v \in G(x^*)$. Since, $\nabla h_i(x^*)$ and $-\nabla h_i(x^*)$belongs to $G(x^*)$, $\forall i = 1, \ldots, m$, it implies that $d^t \nabla h_i(x^*) = 0$. Similarly, since, $\nabla g_j(x^*) \in G(x^*)$, $\forall\, j \in \mathcal{A}(x^*)$, it implies that $d^t \nabla g_j(x^*) \leq 0$.

We discuss the linear approximation of the feasible set with the concept of tangent direction.

Definition 10 (Tangents): A vector d is called a tangent to the feasible set Ω at x^* if there exists a feasible sequence $z_k \to x^*$, and a nonnegative sequence $t_k \to 0$ such that

$$\lim_{k \to \infty} \frac{z_k - x^*}{t_k} = d$$

(9)

i.e., there exists $z_k \to x^*$ such that $\lim_{k \to \infty} \frac{z_k - x^*}{\|z_k - x^*\|} = d$.

Definition 11 (Tangent Cone): Tangent cone denoted by $T_\Omega(x^*)$ is the set of all tangents d.

Lemma 5: For any $x^* \in \Omega$, $T_\Omega(x^*)$ is closed

Proof: See Rodrigo [11]

Lemma 6: For any $x^* \in \Omega$, $T_\Omega(x^*) \subseteq \mathcal{L}(x^*)$.

Proof: Let $d \in T_\Omega(x^*)$, and let $\{z_k\}$, $\{t_k\}$ that give d:

$$z_k = x^* = t_k d + 0(t_k)$$

(10)

We want to show that $d \in \mathcal{L}(x^*)$

From the smoothness of h and g,

$$0 = h(z_k) = h(x^*) + t_k \nabla h(x^*)^t d + 0(t_k)$$

(11)

$$0 = \frac{1}{t_k} h(z_k) = \frac{1}{t_k}(h(x^*) + t_k \nabla h(x^*)^t d + 0(t_k)) = \nabla h(x^*)^t d + \frac{0(t_k)}{t_k} \quad (12)$$

implying that $\nabla h(x^*)^t d = 0$

Similarly,

$$0 \geq g_j(z_k) = g_j(x^*) + t_k \nabla g_j(x^*)^t d + 0(t_k) \quad (13)$$

$$0 \geq \frac{1}{t_k} g_j(z_k) = \frac{1}{t_k}(g_j(x^*) + t_k \nabla g_j(x^*)^t d + 0(t_k)) = \nabla g_j(x^*)^t d + \frac{0(t_k)}{t_k}$$

(14)

implying also that $\nabla g_j(x^*)^t d \leq 0 j \in \mathcal{A}(x^*)$

Thus, $d \in \mathcal{L}(x^*)$.

The converse of the above lemma is false from the following counter example;

Example 2: Consider the function $h: \mathbb{R}^2 \to \mathbb{R}$ and $g: \mathbb{R}^2 \to \mathbb{R}$ defined by

$h(x) = x_1 x_2$ and $g(x) = -x_1 - x_2$ and the points $x^* = (0,0)^t$. We have

$T_\Omega(x^*) = \{d \in \mathbb{R}^2 : d_1 \geq 0, d_2 \geq 0, d_1 d_2 = 0\}$ and $\mathcal{L}(x^*) = \{ d \in \mathbb{R}^2 : -d_1 - d_2 \leq 0\}$.

Thus, $T_\Omega(x^*) \neq \mathcal{L}(x^*)$.

OPTIMALITY CONDITIONS AND CONSTRAINT QUALIFICATIONS

We prove the KKT theorem employing the weakest constraint qualification and discuss other ones that can be easily verified.

Lemma 7 (Fundamental Necessary Condition): If x^* is a local minimum of problem (1), then

$$\nabla f(x^*)^t d \geq 0 \; for all d \in T_\Omega(x^*)$$

Proof: Suppose there exist $d \in T_\Omega(x^*)$ with $\nabla f(x^*)^t d < 0$. Choose; $\{z_k\}$, $\{t_k\}$ that give the limiting direction d. By the Extended Mean Value Theorem,

$$f(z_k) = f(x^*) + (z_k - x^*)^t \nabla f(x^*) + 0(z_k - x^*) \quad (15)$$

$0 > f(z_k) - f(x^*) = t_k d^t \nabla f(x^*) + 0(t_k)$

which yields
$f(z_k) < f(x^*)$
for all t_k sufficiently small. This is a contradiction by the local minimum of x^*.

RESULTS

Theorem 2: (Karush-Kuhn-Tucker (KKT) Conditions in terms GCQ): Let $x^* \in \Omega$ be a local minimum of (1) such that the Guignard Constraint Qualification (GCQ) holds at x^*. Then, there exist $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ such that
$\mu^* \geq 0$

$$\nabla f(x^*)^t + \lambda^* \nabla h(x^*)^t + \mu^* \nabla g(x^*)^t = 0 \qquad (16)$$

$\mu^{*t} g(x^*) = 0$

Proof: Suppose that $x^* \in \Omega$ is the local minimum point of problem (1). By the Lemma 7, we have
$-\nabla f(x^*)^t d \leq 0 \, for all \, d \in T_\Omega(x^*)$.
Using the hypothesis and Lemma 4, we obtain $-\nabla f(x^*) \in (T_\Omega(x^*))^0 = (\mathcal{L}(x^*))^0 = G(x^*)$.
(17)
Thus, there exist $\lambda \in \mathbb{R}^m$ and $\mu_j \geq 0, j \in \mathcal{A}(x^*)$, such that
$-\nabla f(x^*) = \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{j \in \mathcal{A}(x^*)} \mu_j \nabla g_j(x^*)$
(18)

Defining $\lambda^* = \lambda$ and $\mu_j^* = \begin{cases} \mu_j, & for j \in \mathcal{A}(x^*) \\ 0, & otherwise \end{cases}$.

Quasi-regularity Constraint Qualification
The Quasi-regularity Constraint Qualifications also called Abadie Constraint Qualification (ACQ) holds at $x^* \in \Omega$ when $T_\Omega(x^*) = \mathcal{L}(x^*)$. We note that this condition trivially implies that, $(T_\Omega(x^*))^0 = (\mathcal{L}(x^*))^0$. (19)

Example below shows that the above conditions are not equivalent.
Example 3: Consider the function $h: \mathbb{R}^2 \to \mathbb{R}$ and $g: \mathbb{R}^2 \to \mathbb{R}$ defined by

$h(x) = x_1 x_2$ and $g(x) = -x_1 - x_2$ and the points $x^* = (0,0)^t$.
$T_\Omega(x^*) = \{d \in \mathbb{R}^2 : d_1 \geq 0, d_2 \geq 0, d_1 d_2 = 0\} \neq \mathcal{L}(x^*) = \{ d \in \mathbb{R}^2 : -d_1 - d_2 \leq 0\}$. But
$(T_\Omega(x^*))^0 = (\mathcal{L}(x^*))^0 = \{d \in \mathbb{R}^2 : d_1 \leq 0, d_2 \leq 0\}$.

We define some other useful constraint qualifications;
Definition 12: Linear Independence Constraint Qualifications (LICQ): Suppose that $x^* \in \Omega$, and its active sets $h(x^*)$ and $g_j(x^*)$, $j \in \mathcal{A}(x^*)$ the Linear Independence Constraint Qualifications (LICQ) holds if the gradient vectors $\nabla h_i(x^*)^t$ and $\nabla g_j(x^*)^t$, $j \in \mathcal{A}(x^*)$ are linearly Independent.
Definition 13: Mangasarian-Fromovitz's Constraint Qualifications (MFCQ): The Mangasarian-Fromovitz's Constraint Qualifications (MFCQ) holds at x^* when the equality constraint gradients are linearly independent and there exist a vector d such that

$$\nabla h(x^*)^t d = 0 \; and \nabla g_j(x^*)^t d < 0 \; for all j \in \mathcal{A}(x^*)$$

Definition 14: Slater's Constraint Qualifications: Consider the nonlinear optimization problem (1), the Slater's constraint qualifications holds if h is linear, g is convex and there exists $\tilde{x} \in \Omega$ such that

$$h(\tilde{x}) = 0 \ and g(\tilde{x}) < 0.$$

Relationships between Constraint Qualifications

For us to establish our main theorems on the relationship between the constraint qualifications given above, we state the following lemma:

Lemma 8: Let $x^* \in \Omega$ such that the Mangasarian-Fromovitz's Constraint Qualifications (MFCQ) holds at x^*. Then, there exist $\varepsilon \geq 0$ and a C^1-curve $x: (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $x(t) \in \Omega$ for all $t \in [0, \varepsilon)$, $x(0) = x^*$ and $\dot{x}(0) = d$.

Proof: Let $H: \mathbb{R}^n \to \mathbb{R}^m$ be given by $H_i(y,t) = h_i(x^* + td + \mathrm{h}'(x^*)^T y)$ for all $i = 1, \dots, m$ where $\mathrm{h}'(x^*)$ denotes the Jacobian of h at x^*. The nonlinear equation $H(y,t) = 0$ has the solution $(y^*, t^*) = (0,0)$ with $H_y'(0,0) = \mathrm{h}'(x^*)\mathrm{h}'(x^*)^T$ and the latter matrix is non-singular (even positive definite) due to the linear independence of the vectors $\nabla h_i(x^*) (i = 1, \dots, m)$. The implicit function yields a C^1-function $y: (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ such that $y(0) = 0$, $H(y(t), t) = 0$ and $y'(t) = -H_y'(y(t), t)^{-1} H_y'(y(t), t)$ for all $t \in (-\varepsilon, \varepsilon)$. Hence, we have $y'(0) = -H_y'(0,0)^{-1} H_t'(0,0) = -H_y'(0,0)^{-1} \mathrm{h}'(x^*) d = 0$.

Again, put $x(t) = x^* + td + \mathrm{h}'(x^*)^T y(t)$ for all $t \in (-\varepsilon, \varepsilon)$. Reducing ε if necessary, $x: (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ has all the desired properties: Obviously, $x \in C^1$, $x(0) = x^*$, $x'(0) = d$ and $h_i(x(t)) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Furthermore, by continuity $g_j(x(t)) < 0$ for all $j \notin \mathcal{A}(x^*)$ and $|t|$ sufficiently small For $j \in \mathcal{A}(x^*)$ we have $g_j(x(0)) = g_j(x^*) = 0$ and $\frac{d}{dt} g_j(x(0)) = \nabla g_j(x^*)^T d < 0$ and hence $g_j(x(t)) < 0$ for all $t > 0$ sufficiently small.

Relationship between the LICQ and MFCQ

Theorem 4: Suppose that $x^* \in \Omega$ satisfies the Linear Independence Constraint Qualifications (LICQ), then $x^* \in \Omega$ also satisfy the Mangasarian-Fromovitz's Constraint Qualifications (MFCQ).

Proof: Suppose without loss of generality that $\mathcal{A}(x^*) = \{1, \dots, q\}$. Consider the matrix $M = (\nabla h_1(x^*) \dots \nabla h_m(x^*) \nabla g_1(x^*) \dots \nabla g_q(x^*))^{\mathrm{T}}$ and $b \in \mathbb{R}^{m+q}$ given by $b_i = 0$, for all $i = 1, \dots, m$ and $b_j = -1$, for all $j \in \{m+1, \dots, m+q\}$. Since the rows of M are linearly independent, the system $Md = b$ has a solution. Let d^* be a solution. Then, $\nabla h(x^*)^t d^* = 0$ and $\nabla g_j(x^*) d^* = -1 < 0$, for all $j \in \mathcal{A}(x^*)$.

Relationship between MFCQ and ACQ

Theorem 5 If $x^* \in \Omega$ satisfies the Mangasarian-Fromovitz's Constraint Qualifications (MFCQ) then $x^* \in \Omega$ satisfies the Abadie Constraint Qualification (ACQ).

Proof: From **Lemma 6,** recall that for any $x^* \in \Omega$,$T_\Omega(x^*) \subseteq \mathcal{L}(x^*)$. Let $d \in \mathcal{L}(x^*)$ and $\hat{d}$ given by MFCQ (x^*) such that

$\nabla g_j(x^*)^T \hat{d} < 0$ for all $j \in \mathcal{A}(x^*)$,

$\nabla h_i(x^*)^T \hat{d} = 0 \ \ \forall i$

.Put $d(\delta) := d + \delta\hat{d}$ for $\delta > 0$. Then for all $\delta > 0$ we have $\nabla g_j(x^*)^T d(\delta) < 0$ for all $j \in \mathcal{A}(x^*)$, $\nabla h_i(x^*)^T d(\delta) = 0$ for all i.

We claim that this implies $d(\delta) \in T_\Omega(x^*)$ for all $\delta > 0$: By **Lemma 7,** there exists a C^1-curve $x: (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $x(t) \in \Omega$ for all $t \in [0, \varepsilon)$, $x(0) = x^*$ and $\dot{x}(0) = d(\delta)$. For an arbitrary sequence $t_k \to 0$ and z_k we hence infer that $z_k \to x^*$ and thus

$$d(\delta) = \dot{x}(0) = \lim_{k \to \infty} \frac{z_k - x^*}{t_k} \in T_\Omega(x^*) \qquad (20)$$

And since $T_\Omega(x^*)$ is closed, this implies

$$d = \lim_{\delta \to 0} d(\delta) \in T_\Omega(x^*).$$

Relationship between ACQ and GCQ

Theorem 6 If $x^* \in \Omega$ satisfies the Abadie Constraint Qualification (ACQ), then, $x^* \in \Omega$ satisfies the Guignard Constraint Qualification (GCQ).

Proof: This follows immediately from the definition.

Relationship between Slater's CQ and ACQ

Theorem 7: If Slater's condition holds, then $T_\Omega(x^*) = \mathcal{L}(x^*)$ for all, $x^* \in \Omega$.

Proof: Using **Lemma 6,** it suffices to prove that$\mathcal{L}(x^*) \subset T_\Omega(x^*)$. Consider an arbitrary direction $d \in \mathcal{L}(x^*)$ and $\tilde{x} \in \Omega$ given by the Slater condition. Define, $d^* = \tilde{x} - x^*$. By the convexity of g_j we have

$0 > g_j(\tilde{x}) \geq g_j(x^*) + \nabla g_j(x^*)^T d^*$Thus, for $j \in \mathcal{A}(x^*)$, $\nabla g_j(x^*)^T d^* < 0$. Given $\lambda \in (0,1]$,define $\hat{d} = (1-\lambda)d + \lambda d^*$.

We want to prove $\hat{d} \in T_\Omega(x^*)$ for all $\lambda \in (0,1)$.

For $j \in \mathcal{A}(x^*)$, we have $\nabla g_j(x^*)^T d \leq 0$ and $\nabla g_j(x^*)^T d^*$. Consequently, $\nabla g_j(x^*)^T \hat{d} < 0$. Therefore, there exists $\hat{d} = x^* + t\hat{d}$, with $t > 0$ such that $g_j(\hat{x}) < g_j(x^*) = 0$. Taking a sequence, $\{t_k\}$, with $t_k > 0$ and $t_k \longrightarrow 0$, define

$$x^k = (1 - t_k)x^* + t_k\hat{x} = x^* + t_k t\hat{d}.$$

(21)

$$\text{Thus,} \frac{x^k - x^*}{\|x^k - x^*\|} = \frac{t_k t\hat{d}}{\|t_k t\hat{d}\|} = \frac{\hat{d}}{\|\hat{d}\|} \qquad (22)$$

For $\in \mathcal{A}(x^*), g(x^*) < 0$. By the continuity of g, $g(x^k) \leq 0$ for all k sufficiently large. To conclude that $\hat{d} \in T_\Omega(x^*)$ it is enough to show $h(x^k) = 0$ for all k sufficiently large

Since $d \in \mathcal{L}(x^*)$, $Md = \nabla \mathrm{h}(\mathrm{x})\mathrm{d} = 0$. Furthermore, $Md^* = M(\tilde{x} - x^*) = 0$. Consequently, $M\hat{d} = 0$. Thus, $h(x^k) = Mx^k - c = Mx^* - c + t_k tM\hat{d} = 0$, since $x^* \in \Omega$. So, $\hat{d} \in T_\Omega(x^*)$, which implies $d \in T_\Omega(x^*)$, since $T_\Omega(x^*)$ is a closed set.

CONCLUSION

In conclusion, we observe that the Guignard Constraint Qualification (GCQ), $(T_\Omega(x^*))^0 = (\mathcal{L}(x^*))^0$ which follows trivially from Abadie Constraint Qualification (ACQ), $x^* \in \Omega$ $T_\Omega(x^*) = \mathcal{L}(x^*)$ is the weakest constraint qualification while the Linear Independence Constraint Qualifications (LICQ) is the strongest constraint qualification at $x^* \in \Omega$.

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