International Digital Organization for Scientific Research ISSN: 2550-794X
IDOSR JOURNAL OF SCIENTIFIC RESEARCH 3(3) 1-12, 2018.

A Mathematical Model for the Control of Cholera without Natural Recovery

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ABSTRACT

We present in this research work, a mathematical model for the control of cholera without natural recovery. This follows a slight modification as compared to previous cholera models. Our model incorporates treatment, water hygiene as well as environmental sanitation in an attempt to curtailing the disease. We employ a system of nonlinear ordinary differential equations for our model. The model is analyzed and shows that with proper combination of control measures the spread of cholera could be reduced. Simulation of the full model using MathCAD shows that improvement in treatment, water hygiene and the environmental waste management is indeed effective in eradicating the cholera epidemic.

Keywords: Stability, Simulation, Mathematical model.

INTRODUCTION

Cholera is a contagious infectious disease that is characterized by extreme vomiting, profuse watery diarrhea and leg pain. It has been found that transmission transpires mostly via absorption of contaminated drinking water or food. Worldwide, almost every year there is an estimated 3-5 million cholera cases and 100, 000-130, 000 deaths due to cholera a year as of 2010[15]. It has a very short incubation period which starts from a few hours to five days. The health of an infected person disintegrates rapidly and death may occur if treatment is not promptly given. Cholera was first discovered in the Indian subcontinent in 1817. The disease reaches all the way through Asian continent in the 1960s, getting in to Africa in 1970 and Latin America in 1991 [4]. In many parts of Africa and Asia the disease is still endemic. Cholera is a disastrous water-borne infectious disease that is caused by the bacterium vibrio cholera. It is a very serious problem in many developing countries due to inadequate access to safe drinking water supply, improper treatment of reservoirs and improper sanitation. In 2012, WHO reported 245, 393 cholera cases and 3034 death cases across 48 countries in which 67% cases occurred in African countries [16]. In 2005, Nigeria had 4, 477 cases and 174 deaths. There were reported cases of cholera in 2008 in Nigeria in which there were 429 deaths out of 6, 330 cases. Furthermore, 2, 304 cases were reported in Niger State in which 114 were death cases [7]. Also in 2009, Nigeria reported 13, 691 cases and 431 deaths

[14]. [12][11] evidenced that recent years have seen a strong trend of cholera outbreak in developing countries, such as in India (2007), Iraq (2008), Congo (2008), Zimbabwe (2008-2009), Haiti (2010), Kenya (2010) and Nigeria (2010).

In Nigeria, outbreaks of the disease have been taking place with ever-increasing occurrence ever since the earliest outbreak in recent times in 1970, and more recently there were outbreaks of Cholera in the Northern parts of the country(WHO 2018) [5]. In summary the United Nation (UN) unit, reports: "despite Nigeria's oil wealth, more than 70% of the country's 126 million people live below the poverty line and cholera outbreaks are common in poor urban areas which lack proper sanitation and clean drinking water" (UN Office for the Coordination of Humanitarian Affairs Integrated Regional Information Networks (IFIN) 2005). In the last few decades,[2],[4],[10], [16],[6],[1],[13],[18],[12] and [9], have designed mathematical models to explore the transmission dynamics and control of the disease. However, there have not been many studies on cholera in Nigeria using mathematical modeling. Hence we decided to apply mathematical model to study the disease without natural recovery.

Model formulation

A mathematical model for cholera transmission is developed by making some relevant modofications to the previous models [2]. Our model is devoid of natural recovery, this is because cholera is a fatal disease, and a large percentage of those who recover, do so because of treatment. Our model on the other hand incorporates treatment, water hygiene and environmental waste management. The model contains five variables which are susceptible, infected, recovered, total

human population, and the concentration of vibrio cholera in water. The susceptible population is generated either through birth or through immigration. They acquire infection and move to the infected class at the rate:

$$\frac{\alpha\left(1-c_w\right)B}{K_1+K_2}$$

a = The rate of exposure to contaminated water

c_w = The rate of compliance with water hygiene

K_1 = The concentration of vibrio cholerae in water that yields 50% chance of catching cholera (Codecco, 2001)

K_2 = The concentration of vibrio cholerae in contaminated water

The number of infected individuals decreases through recovery from the disease, at the rate τ being recovery due to treatment. Bacterial populations in the aquatic environment grow in the water at the rate determined by the environmental factors such as temperature. A schematic depiction of the above explanation is shown in Fig. 1 and the variables and parameters of the model are described in Figure 1: A schematic representation of our model is given below: The model consists of the following system of ordinary differential equations given in (1) to (4):

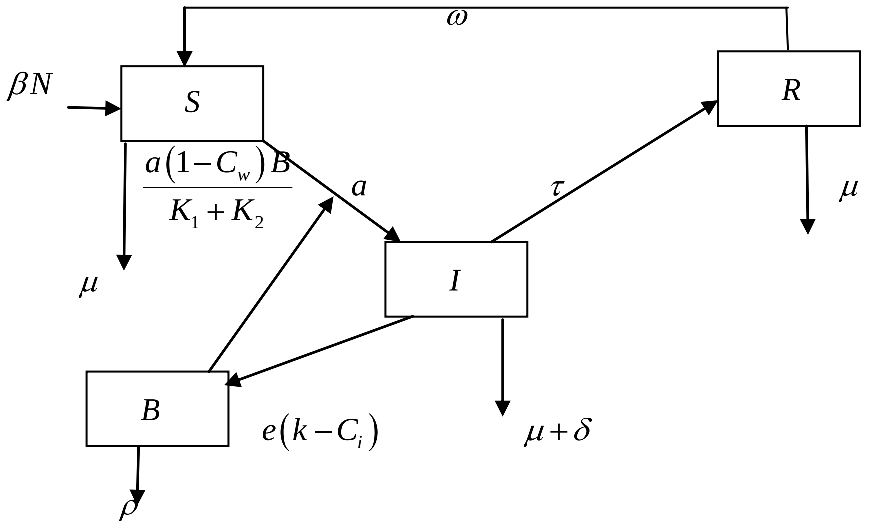

Fig.1 Schematic diagram of controlled cholera transmission model without natural recovery.

Table-1 Variables of the cholera model

Symbol	Description
$S(t)$	Susceptible human population at time t
$I(t)$	Infected human population at time t
$R(t)$	Recovered human population at time t
$B(t)$	Concentration of vibrio cholera population at time t
N	Total population of humans

Table-2 Parameters of the cholera model

Symbol	Description
β	Per capita birth rate of humans
μ	Per capita death rate of humans
a	Rate of exposure to contaminated water
K_1	Concentration of vibrio cholera in water
τ	Recovery rate due to treatment
c_w	Rate of compliance with water hygiene
c_i	Rate of compliance with environmental sanitation
δ	Cholera induced death
ρ	Growth rate of vibrio cholera in the aquatic environment
ω	Loss rate of immunity by recovered individuals
e	Contribution of each infected person to the population of vibrio cholera in the aquatic environment
K_2	Concentration of vibrio cholerae in contaminated water

The model consists of the following system of ordinary differential equations given in (1) to (4):

$$\frac{dS}{dt}=\beta N-\frac{a(1-c_w)B}{K_1+K_2}S+\omega R-\mu S \qquad 1$$

$$\frac{dI}{dt}=\frac{a(1-c_w)B}{K_1+K_2}S-(\tau+\mu+\delta)I \qquad 2$$

$$\frac{dB}{dt}=e(k-c_i)I-\rho B \qquad 3$$

$$\frac{dR}{dt}=\tau I-(\omega+\mu)R \qquad 4$$

Epidemic dynamics

The disease-free equilibrium (DFE) for the model is given by

Basic Properties of the Model

Positivity of Solutions

Since the model monitors human population, we need to show that all the state variables remain non-negative for all times.

Theorem

Let $\Omega=\left\{(S,I,R,B)\in\Re_+^4 : S(0)>0, I(0)>0, R(0)>0, B(0)>0\right\}$ then the solutions of $\left\{S(t),I(t),R(t),B(t)\right\}$ of the system equation (1-4) are positive for all $t\geq 0$.

Proof:

Consider the first equation of the system eqn.

$$\frac{dS}{dt}=\beta N-\frac{a(1-c_w)B}{K_1+K_2}S+\omega R-\mu S \qquad 5$$

$$\Rightarrow \quad \frac{dS}{dt}\geq \beta N-\mu S \qquad 6$$

We consider the solution of

$$\frac{dS}{dt}+\mu S=\beta N \qquad 7$$

Employing integrating factor method, i.e.

$$I(t)=e^{\int \mu dt}=e^{\mu t} \qquad 8$$

Multiplying the ODE through by the integrating factor, we have

$$e^{\mu t}\frac{dS}{dt}+\mu e^{\mu t}S=\beta N e^{\mu t} \qquad 9$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{\mu t}S\right)=\beta N e^{\mu t} \qquad 10$$

$$\Rightarrow \quad e^{\mu t}S=\int \beta N e^{\mu t}dt+C \qquad 11$$

$$\therefore \quad e^{\mu t}S=\frac{\beta N}{\mu}e^{\mu t}+C$$ 12 (C = constant of integration)

$$\Rightarrow \quad S(t)=\frac{\beta N}{\mu}+Ce^{-\mu t} \qquad 13$$

With the initial condition

$$S(0)=S_0 \quad \Rightarrow \quad C=S_0-\frac{\beta N}{\mu} \qquad 14$$

$$\Rightarrow \quad S(t)=\frac{\beta N}{\mu}+\left(S_0-\frac{\beta N}{\mu}\right)e^{-\mu t}\geq 0 \qquad 15$$

We now consider the second equation;

$$\frac{dI}{dt}=\frac{a(1-c_w)B}{K_1+K_2}S-(\tau+\mu+\delta)I \qquad 16$$

$$\therefore \quad \frac{dI}{dt}\geq-(\tau+\mu+\delta)I \qquad 17$$

Integration yields

$$I(t)\geq I_0e^{-(\tau+\mu+\delta)t}\geq 0 \qquad 18$$

We consider also the third equation

$$\frac{dB}{dt}=e(K-c_i)I-\rho B \qquad 19$$

$$\therefore \quad \frac{dB}{dt}\geq-\rho B \qquad 20$$

$$\Rightarrow \quad B(t)\geq B_0e^{-\rho Bt} \qquad 21$$

We observe that as $t\to\infty, B(t)\to B_0$.

Finally from the fourth equation

$$\frac{dR}{dt}=\tau I-(\omega+\mu)R \qquad 22$$

$$\frac{dR}{dt}\geq-(\omega+\mu)R \qquad 23$$

$$\Rightarrow \quad R(t)\geq R_0e^{-(\omega+\mu)t}\geq 0 \qquad 24$$

Invariant Region

The system equations (1-4) have solutions which are contained in the feasible region:

$$\Omega=\left\{(S,I,R,B)\in\Re_+^4:0\leq S(0),0\leq I(0),0\leq R(0),0\leq B(0),S+I+R\leq\frac{\beta N}{\mu}\right\}$$

where the model makes biological sense can be shown to be positively invariant and globally attracting in $\Re_+^4$ with respect to the system of ODE describing our model.

Disease Free Equilibrium (DFE)

The disease free equilibrium of model system (1-4) is obtained by setting $\frac{dS}{dt}=\frac{dI}{dt}=\frac{dB}{dt}=\frac{dR}{dt}=0$, and in the absence of disease, $I=B=0$ so that:

$$\beta N-\frac{a\left(1-c_w\right)B}{K_1+K_2}S+\omega R-\mu S=0 \qquad 25$$

$$\frac{a\left(1-c_w\right)B}{K_1+K_2}S-\left(\tau+\mu+\delta\right)I=0 \qquad 26$$

$$e\left(k-c_i\right)I-\rho B=0 \qquad 27$$

$$\tau I-\left(\omega+\mu\right)R=0 \qquad 28$$

Since $I=B=0$, equation (4) gives $R=0$ and equation (1) gives $S=\dfrac{\beta N}{\mu}$.

Hence DFE is

$$\left(S^0,I^0,B^0,R^0\right)=\left(\frac{\beta N}{\mu},0,0,0\right) \qquad 29$$

Model Analysis

The model system (1-4) is analyzed qualitatively to get insights into its dynamical features which give better understanding of the impact control strategies on the transmission dynamics of the vibrio cholera disease.

Stability Analysis

$$J\left(S,I,B,R\right)=\begin{pmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial B} & \frac{\partial f_1}{\partial R} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial B} & \frac{\partial f_2}{\partial R} \\ \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial B} & \frac{\partial f_3}{\partial R} \\ \frac{\partial f_4}{\partial S} & \frac{\partial f_4}{\partial I} & \frac{\partial f_4}{\partial B} & \frac{\partial f_4}{\partial R} \end{pmatrix}$$

where

$$f_1\left(S,I,B,R\right)=\beta N-\frac{a\left(1-c_w\right)B}{K_1+K_2}S+\omega R-\mu S \qquad 30$$

$$f_2\left(S,I,B,R\right)=\frac{a\left(1-c_w\right)B}{K_1+K_2}S-\left(\tau+\mu+\delta\right)I \qquad 31$$

$$f_3\left(S,I,B,R\right)=e\left(k-c_i\right)I-\rho B \qquad 32$$

$$f_4\left(S,I,B,R\right)=\tau I-\left(\omega+\mu\right)R \qquad 33$$

From the above we have the following partial derivatives.

$$\frac{\partial f_1}{\partial S}=-\frac{a(1-c_w)B}{K_1+K_2}-\mu,\ \frac{\partial f_1}{\partial I}=0,\ \frac{\partial f_1}{\partial B}=-\frac{aK(1-c_w)}{(K_1+K_2)^2},\ \frac{\partial f_1}{\partial R}=\omega$$

$$\frac{\partial f_2}{\partial S}=\frac{a(1-c_w)B}{K_1+K_2},\ \frac{\partial f_2}{\partial I}=-(\tau+\mu+\delta),\ \frac{\partial f_2}{\partial B}=\frac{aK(1-c_w)}{(K_1+K_2)^2},\ \frac{\partial f_2}{\partial R}=0$$

$$\frac{\partial f_3}{\partial S}=0,\ \frac{\partial f_3}{\partial I}=e(k-c_i),\ \frac{\partial f_3}{\partial B}=-\rho,\ \frac{\partial f_3}{\partial R}=0$$

$$\frac{\partial f_4}{\partial S}=0,\ \frac{\partial f_4}{\partial I}=\tau,\ \frac{\partial f_4}{\partial B}=0,\ \frac{\partial f_4}{\partial R}=-(\omega+\mu)$$

$$\Rightarrow \quad J(S,I,B,R)=\begin{pmatrix} -\frac{a(1-c_w)B}{K_1+k_2}-\mu & 0 & -\frac{aK(1-c_w)}{(K_1+K_2)^2} & \omega \\ \frac{a(1-c_w)B}{K_1+K_2} & -(\tau+\mu+\delta) & \frac{aK(1-c_w)}{(K_1+K_2)^2} & 0 \\ 0 & e(k-c_i) & -\rho & 0 \\ 0 & \tau & 0 & -(\omega+\mu) \end{pmatrix} \qquad 34$$

And the Jacobian at the disease free equilibrium state $(S^0,I^0,B^0,R^0)=\left(\frac{\beta N}{\mu},0,0,0\right)$ is

$$J^*(S^0,I^0,B^0,R^0)=\begin{pmatrix} -\mu & 0 & -\frac{a(1-c_w)}{K} & \omega \\ 0 & -(\tau+\mu+\delta) & \frac{a(1-c_w)}{K} & 0 \\ 0 & e(k-c_i) & -\rho & 0 \\ 0 & \tau & 0 & -(\omega+\mu) \end{pmatrix} \qquad 35$$

The eigen-values of J^* are given by

$$\left|J^*-\lambda I\right|=0 \qquad 36$$

$$\begin{aligned}&\lambda^4+(3\mu+\rho+\tau+\delta)K\lambda^3\\ &+\left(3K\mu^2-ae+aec_s+aec_w+3K\mu\rho+2K\mu\tau+K\rho\tau+2K\mu\delta+K\rho\omega+K\rho\delta+K\tau\omega+K\omega\delta-aec_sc_w\right)\lambda^2\\ &+\begin{pmatrix} K\mu^3+3K\mu^2\rho+K\mu^2\tau+K\mu^2\omega+K\mu^2\delta-2ae\mu-ae\omega+2ae\mu c_s+2ae\mu c_w+ae\omega c_s+ae\omega c_w \\ +2K\mu\rho\tau+2K\mu\rho\omega+2K\mu\rho\delta+K\mu\tau\omega+K\rho\tau\omega+K\mu\omega\delta+K\rho\omega\delta-2ae\mu c_sc_w-ae\omega c_sc_w \end{pmatrix}\lambda\\ &+\end{aligned}$$

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0$$ 37

The parameter values in the Tables 1 and 2 are estimated.

Table 3: Values for population-dependent parameters of the model

S/NO	Variable/Parameter	Value	Source
1	S	82,104,841	CIA [17]
2	R	37,120,961	CIA [17]
3	I	5,792,995	CIA [17]
4	B	3,348,245	CIA [17]
5	N	177,155,754	CIA [17]

Table 4: parameter values used in the cholera model

S/NO	Parameter	Value	Source
1	β	0.0000621 yr^{-1}	WHO [18]
2	μ	0.0000526 day^{-1}	CIA [18]
3	ω	0.5 yr^{-1}	CIA [17]
4	a	0.1 yr^{-1}	CIA [17]
5	δ	0.00292 yr^{-1}	CDC [19]
6	τ	1.25 yr^{-1}	CDC [19]
7	K	10^6	Codecco
8	ℓ	10cells/mL/day	Codecco
7	K_1, k_2, v, c_i, c_w,	(0-1) yr^{-1}	Estimated

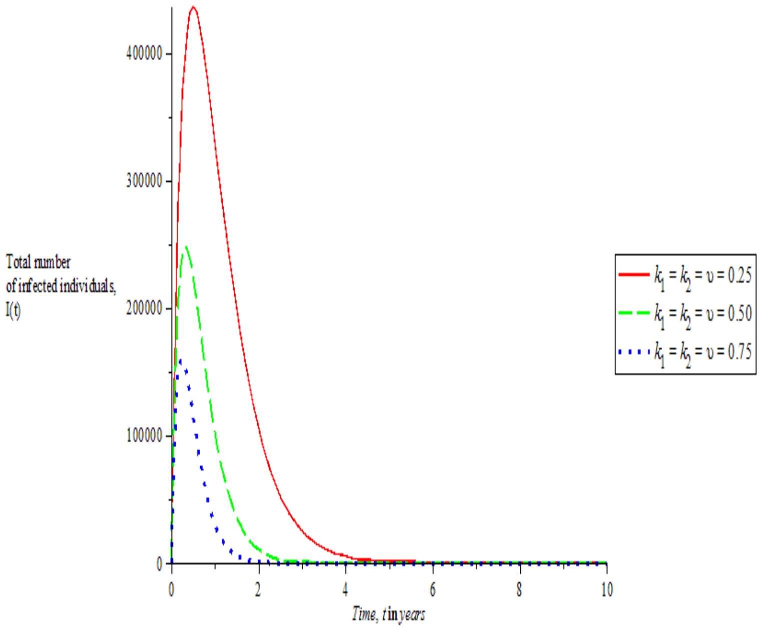

Figure 2: Death rate due to Cholera with no treatment. Parameter values used are displayed in Table 4 above.

We observe from figure 2 that the death rate due to Cholera increases without treatment. This agrees with our intuitive reasoning in the sense that, if cholera is left untreated for a long time (i.e. if the infection has reached its maximum a) an individual will die of the disease no matter the control measures put in place.

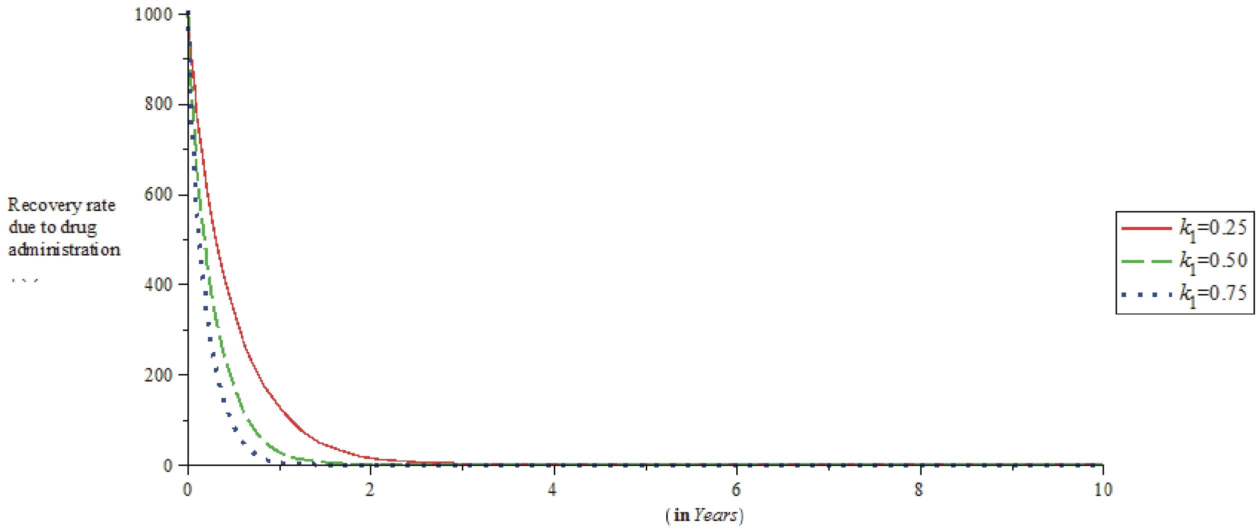

Figure 3: Recovery rates due to drug administration.

We observe from figure 4 that as the infection increases, the recovery rates slows down. This simply implies that, the lower the infection the faster the recovery rate and the higher the infection the slower the rate of recovery. The results of figure 3 agree with reality as obtainable in health centers.

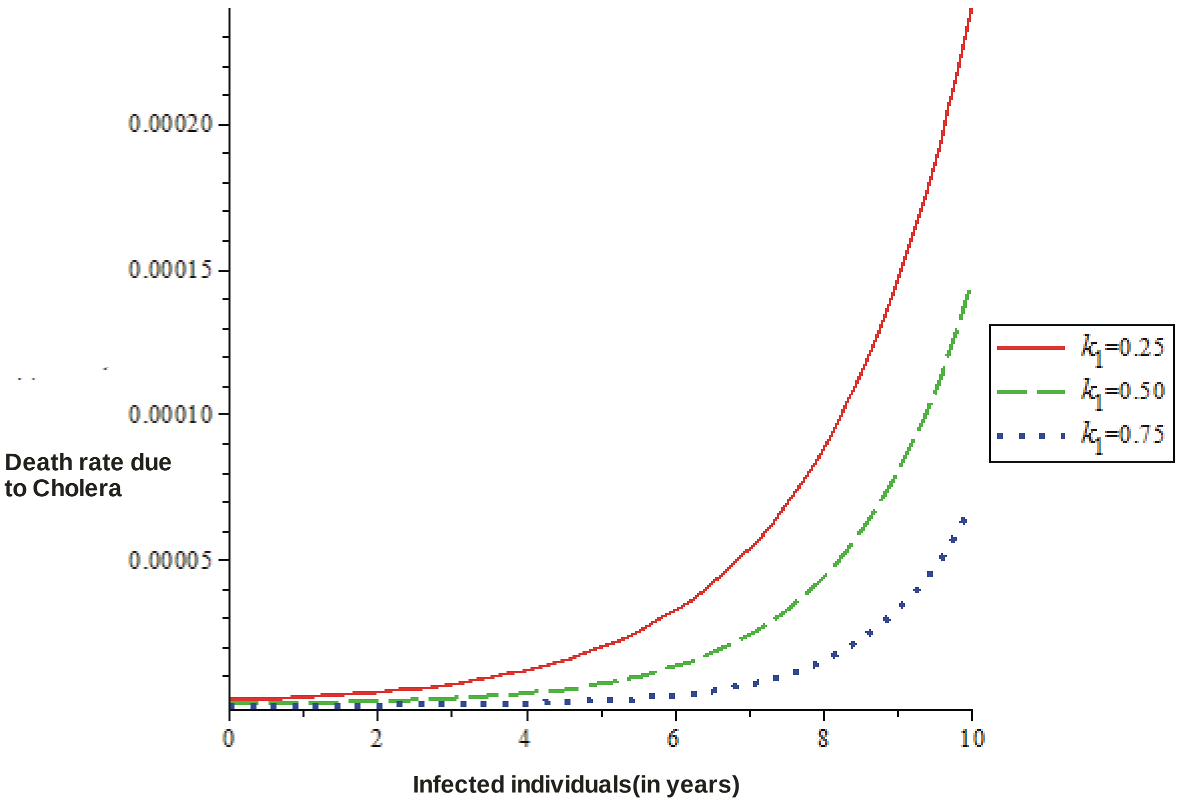

Figure 4: Total numbers of infected individuals against time, t. Parameter values are as shown in Table 3 and 4 above.

Figure 4 shows the comparison between low, moderate and high control measures as they affect the total number of actively infected individuals. It is observed that as the treatment level increases, the total number of actively infected individuals with Cholera decreases. The reduction is however achievable in less than 2 years with 80% treatment level.

RESULTS

The model is simulated using the parameter values in Table 4 to assess the effect of the three main control measures considered in this study, namely, treatment, rate of compliance with water hygiene and rate of compliance with environmental sanitation $\left(\tau, C_w \text{ and } C_s\right)$.

CONCLUSION

In this study, we presented an improved mathematical model for the transmission and control of cholera in Nigeria. The model incorporates treatment, water hygiene and environmental waste management. Different control strategies are used to see the effect of control measures on eradication of cholera. Our study shows that 80% level of control measures is sufficient to effectively control the spread of cholera in Nigeria. The total number of the infected individuals decreases with the increase in level of the control measures stopping the disease from reaching an alarming level.

We observe too that whenever the control strategies are carried out solely then treatment is best alternative to cholera, but when there are two combinations strategies then the best combination is
treatment and water hygiene On the other hand when a combination of three control strategies is implemented then the best combination is the treatment, water hygiene and environmental waste management. From this study we conclude that the more one increases combination of control strategies then cholera can be eradicated from the nation.

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